
Robust approachability and regret minimization in games with partial monitoring

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Abstract

Approachability has become a standard tool in analyzing learning algorithms in the adversarial online learning setup. We develop a variant of approachability for games where there is ambiguity in the obtained reward that belongs to a set, rather than being a single vector. Using this variant we tackle the problem of approachability in games with partial monitoring and develop simple and efficient algorithms for this setup. We finally consider external and internal regret in repeated games with partial monitoring, for which we show efficient regret-minimizing strategies based on approachability theory.

1 Introduction

Blackwell's approachability theory and its variants has become a standard and useful tool in analyzing online learning algorithms (Cesa-Bianchi and Lugosi, 2006) and algorithms for learning in games (Hart and Mas-Colell, 2000, 2001). The first application of Blackwell's approachability to learning in the online setup is due to Blackwell himself in Blackwell (1956b). Numerous other contributions are summarized in Cesa-Bianchi and Lugosi (2006). Blackwell's approachability theory enjoys a clear geometric interpretation that allows it to be used in situations where online convex optimization or exponential weights do not seem to be easily applicable and, in some sense, to go beyond the minimization of the regret and/or to control quantities of a different flavor; e.g., in Mannor et al. (2009), to minimize the regret together with path constraints, and in Mannor and Shimkin (2008), to minimize the regret in games whose stage duration is not fixed. Recently, it has been shown that approachability and low regret learning are equivalent in the sense that efficient reductions exist from one to the other (Abernethy et al., 2010). Another recent paper (Rakhlin et al., 2010) showed that approachability can be analyzed from the perspective of learnability using tools from learning theory.

In this paper we consider approachability and online learning with partial monitoring in games against Nature. In partial monitoring the decision maker does not know how much reward was obtained and only gets a (random) signal whose distribution depends on the action of the decision maker and the action of Nature. There are two extremes of this setup that are well studied. On the one extreme we have the case where the signal includes the reward itself (or a signal that can be used to unbiasedly estimate the reward), which is essentially the celebrated bandits setup. The other extreme is the case where the signal is not informative (i.e., it tells the decision maker nothing about the actual reward obtained); this setting then essentially consists of repeating the same situation over and over again, as no information is gained over time. We consider a setup encompassing these situations and more general ones, in which the signal is indicative of the actual reward, but is not necessarily a sufficient statistics thereof. The difficulty is that the decision maker cannot compute the actual reward he obtained nor the actions of Nature.

Regret minimization with partial monitoring has been studied in several papers in the learning theory community. Piccolboni and Schindelhauer (2001), Mannor and Shimkin (2003), Cesa-Bianchi et al. (2006) study special cases where there is some extra structure that allows to estimate the reward (or worst-case reward) of the decision maker efficiently. A general policy with vanishing regret is presented in Lugosi et al. (2008). This policy is based on exponential weights and a specific estimation procedure for the (worst-case) obtained reward. In contrast, we provide approachability-based results for the problem of regret minimization. On route, we define a new type of approachability setup and extend approachability efficiently to the partial monitoring vector-valued setting.

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The paper is organized as follows. In Section 2 we recall some basic facts from approachability theory. In Section 3 we propose a novel setup for approachability, termed “robust approachability,” where instead of obtaining a vector-valued reward, the decision maker obtains a set, that represents the ambiguity concerning his reward. We provide a simple characterization of approachable convex sets and an efficient algorithm for the set-valued reward setup. In Section 4 we show how to apply the robust approachability framework to the repeated vector-valued games with partial monitoring. We provide a simple and constructive algorithm for this setup. Previous results for approachability in this setup were either non-constructive (Rustichini, 1999) or were inefficient as they relied on some sort of lifting to the space of probability measures on mixed actions (Perchet, 2011) and typically required a grid that is progressively refined. In Section 5 we apply our results for both external and internal regret minimization with partial monitoring. In both cases our proofs are simple, lead to efficient algorithms, and are accompanied with rates. Our results for external regret have similar rates to Lugosi et al. (2008), but our proof is direct and simpler. For internal regret minimization we present the first efficient algorithm and the first convergence rates.

2 Some basic facts from approachability theory

In this section we recall the most basic versions of Blackwell’s approachability theorem for vector-valued payoff functions.

We consider a vector-valued game between two players, with respective finite action sets \mathcal{A} and \mathcal{B} , whose cardinalities are respectively referred to as $N_{\mathcal{A}}$ and $N_{\mathcal{B}}$. We denote by d the dimension of the reward vector and equip \mathbb{R}^d with the ℓ^2 -norm $\|\cdot\|_2$. The payoff function of the first player is given by a mapping $m : \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}^d$, which is multi-linearly extended to $\Delta(\mathcal{A}) \times \Delta(\mathcal{B})$, the set of product-distributions over $\mathcal{A} \times \mathcal{B}$.

We consider two frameworks, depending on whether pure or mixed actions are taken.

Pure actions taken and observed. We denote by A_1, A_2, \dots and B_1, B_2, \dots the actions in \mathcal{A} and \mathcal{B} sequentially taken by each player; they are possibly given by randomized strategies, i.e., the actions A_t and B_t were obtained by random draws according to respective probability distributions denoted by $\mathbf{x}_t \in \Delta(\mathcal{A})$ and $\mathbf{y}_t \in \Delta(\mathcal{B})$. For now, we assume that the first player has a full monitoring of the pure actions taken by the opponent player: at the end of round t , when receiving the payoff $m(A_t, B_t)$, the pure action B_t is revealed to him.

Definition 1 A set $\mathcal{C} \subseteq \mathbb{R}^d$ is m -approachable with pure actions if there exists a strategy¹ of the first player such that for all strategies of the second player,

$$\limsup_{T \rightarrow \infty} \inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T m(A_t, B_t) \right\|_2 = 0 \quad a.s.$$

That is, the first player has a strategy that ensures that the average of his vector-valued payoffs converges to the set \mathcal{C} .

Mixed actions taken and observed. In this case, we denote by $\mathbf{x}_1, \mathbf{x}_2, \dots$ and $\mathbf{y}_1, \mathbf{y}_2, \dots$ the actions in $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$ sequentially taken by each player. We also assume a full monitoring for the first player: at the end of round t , when receiving the payoff $m(\mathbf{x}_t, \mathbf{y}_t)$, the mixed action \mathbf{y}_t is revealed to him.

Definition 2 In this context, a set $\mathcal{C} \subseteq \mathbb{R}^d$ is m -approachable with mixed actions if there exists a strategy of the first player such that for all strategies of the second player,

$$\limsup_{T \rightarrow \infty} \inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T m(\mathbf{x}_t, \mathbf{y}_t) \right\|_2 = 0 \quad a.s.$$

Necessary and sufficient condition for approachability. For closed convex sets there is a simple characterization of approachability that is a direct consequence of the minimax theorem; the condition is the same for the two settings, whether pure or mixed actions are taken and observed.

Theorem 3 (Blackwell 1956a, Theorem 3) A closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$ is approachable (with pure or mixed actions) if and only if

$$\forall \mathbf{y} \in \Delta(\mathcal{B}), \exists \mathbf{x} \in \Delta(\mathcal{A}), \quad m(\mathbf{x}, \mathbf{y}) \in \mathcal{C}.$$

¹The original definition given by Blackwell requires uniformity w.r.t. the strategy set of the opponent. We ignore the uniformity to avoid excessive nomenclature.

In the latter case, the following convergence rates are achieved by an efficient strategy. We denote by M a bound in norm over m , i.e.,

$$\max_{(a,b) \in \mathcal{A} \times \mathcal{B}} \|m(a,b)\|_2 \leq M.$$

With mixed actions taken and observed, for all strategies of the second player, with probability 1,

$$\inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T m(\mathbf{x}_t, \mathbf{y}_t) \right\|_2 \leq \frac{2M}{\sqrt{T}}.$$

With pure actions taken and observed, for all $\delta \in (0, 1)$ and for all strategies of the second player, with probability at least $1 - \delta$,

$$\inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T m(A_t, B_t) \right\|_2 \leq \frac{2M}{\sqrt{T}} \left(1 + 2\sqrt{\ln(2/\delta)}\right).$$

The proof is provided in Appendix B.1 for completeness.

An efficient associated strategy. Blackwell suggested a simple strategy with a geometric flavor. (This strategy can be extracted from the proof of Theorem 3 provided in appendix.)

Play an arbitrary \mathbf{x}_1 . For $t \geq 1$, given the vector-valued quantities

$$\hat{m}_t = \frac{1}{t} \sum_{\tau=1}^t m(\mathbf{x}_\tau, B_\tau) \quad \text{or} \quad \hat{m}_t = \frac{1}{t} \sum_{\tau=1}^t m(\mathbf{x}_\tau, \mathbf{y}_\tau),$$

depending on whether pure or mixed actions are taken and observed, compute the projection c_t (in ℓ^2 -norm) of \hat{m}_t on \mathcal{C} . Find a mixed action \mathbf{x}_{t+1} that solves the minimax equation

$$\min_{\mathbf{x} \in \Delta(\mathcal{A})} \max_{\mathbf{y} \in \Delta(\mathcal{B})} \langle \hat{m}_t - c_t, m(\mathbf{x}, \mathbf{y}) \rangle, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidian inner product in \mathbb{R}^d . The minimax problem above is easily seen to be a (scalar) zero-sum game and is therefore efficiently solvable using, e.g., linear programming.

In the case when pure actions are taken and observed, it only remains to draw A_{t+1} at random according to \mathbf{x}_{t+1} .

3 Robust approachability

In this section we extend the results of the previous section to set-valued payoff functions. To this end, we denote by $\mathcal{S}(\mathbb{R}^d)$ the set of all measurable subsets of \mathbb{R}^d and consider a set-valued payoff function $\bar{m} : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{S}(\mathbb{R}^d)$.

Pure actions taken and observed. At each round t , the players choose simultaneously respective actions $A_t \in \mathcal{A}$ and $B_t \in \mathcal{B}$, possibly at random according to mixed distributions \mathbf{x}_t and \mathbf{y}_t . As a result, the first player gets the subset $\bar{m}(A_t, B_t)$ as a payoff. This models the ambiguity or uncertainty associated with some true underlying payoff gained. Full monitoring still takes place for the first player: he observes B_t at the end of round t .

We extend \bar{m} multi-linearly to $\Delta(\mathcal{A}) \times \Delta(\mathcal{B})$ and even to $\Delta(\mathcal{A} \times \mathcal{B})$, the set of joint probability distributions on $\mathcal{A} \times \mathcal{B}$, as follows. Let

$$\mu = (\mu_{a,b})_{(a,b) \in \mathcal{A} \times \mathcal{B}}$$

be such a joint probability distribution; then $\bar{m}(\mu)$ is defined as a finite convex combination² of subsets of \mathbb{R}^d ,

$$\bar{m}(\mu) = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \mu_{a,b} \bar{m}(a, b).$$

When μ is the product-distribution of some $\mathbf{x} \in \Delta(\mathcal{A})$ and $\mathbf{y} \in \Delta(\mathcal{B})$, we use the notation $\bar{m}(\mu) = \bar{m}(\mathbf{x}, \mathbf{y})$.

We denote by

$$\pi_T = \frac{1}{T} \sum_{t=1}^T \delta_{(A_t, B_t)}$$

²For two sets S , T and $\alpha \in [0, 1]$, the convex combination $\alpha S + (1 - \alpha)T$ is defined as

$$\{\alpha s + (1 - \alpha)t, \quad s \in S \text{ and } t \in T\}.$$

the empirical distribution of the pairs (A_t, B_t) of actions taken during the first T rounds and will be interested in the behavior of

$$\frac{1}{T} \sum_{t=1}^T \bar{m}(A_t, B_t),$$

which can also be rewritten here in a compact way as $\bar{m}(\pi_T)$, by linearity of \bar{m} .

Definition 4 Let $\mathcal{C} \subseteq \mathbb{R}^d$ be some set; \mathcal{C} is \bar{m} -approachable with pure actions if there exists a strategy of the first player such that for all strategies of the second player,

$$\limsup_{T \rightarrow \infty} \sup_{d \in \bar{m}(\pi_T)} \inf_{c \in \mathcal{C}} \|c - d\|_2 = 0 \quad a.s.$$

That is, when \mathcal{C} is \bar{m} -approachable with pure actions, the first player has a strategy that ensures that the average of the sets of payoffs converges to the set \mathcal{C} : the sets $\bar{m}(\pi_T)$ are included in ε_T -neighborhoods of \mathcal{C} , where the sequence of ε_T tends almost-surely to 0.

Mixed actions taken and observed. At each round t , the players choose simultaneously respective mixed actions $\mathbf{x}_t \in \Delta(\mathcal{A})$ and $\mathbf{y}_t \in \Delta(\mathcal{B})$, and the first player gets the subset $\bar{m}(\mathbf{x}_t, \mathbf{y}_t)$ as a payoff. Full monitoring still takes place for the first player: he observes \mathbf{y}_t at the end of round t .

The product-distribution of two elements $\mathbf{x} = (x_a)_{a \in \mathcal{A}} \in \Delta(\mathcal{A})$ and $\mathbf{y} = (y_b)_{b \in \mathcal{B}} \in \Delta(\mathcal{B})$ will be denoted by $\mathbf{x} \otimes \mathbf{y}$; it gives a probability mass of $x_a y_b$ to each pair $(a, b) \in \mathcal{A} \times \mathcal{B}$. We consider the empirical joint distribution of mixed actions taken during the first T rounds,

$$\nu_T = \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \otimes \mathbf{y}_t,$$

and will be interested in the behavior of

$$\frac{1}{T} \sum_{t=1}^T \bar{m}(\mathbf{x}_t, \mathbf{y}_t),$$

which can also be rewritten here in a compact way as $\bar{m}(\nu_T)$, by linearity of \bar{m} .

Definition 5 Let $\mathcal{C} \subseteq \mathbb{R}^d$ be some set; \mathcal{C} is \bar{m} -approachable with mixed actions if there exists a strategy of the first player such that for all strategies of the second player,

$$\limsup_{T \rightarrow \infty} \sup_{d \in \bar{m}(\nu_T)} \inf_{c \in \mathcal{C}} \|c - d\|_2 = 0 \quad a.s.$$

A useful continuity lemma. Before proceeding we provide a continuity lemma. It can be reformulated, with the notation defined in its statement, as indicating that for all joint distributions μ and ν over $\mathcal{A} \times \mathcal{B}$, the set $\bar{m}(\mu)$ is included in a $M \|\mu - \nu\|_1$ -neighborhood of $\bar{m}(\nu)$, a fact that we will use repeatedly below.

Lemma 6 Let μ and ν be two probability distributions over $\mathcal{A} \times \mathcal{B}$. We assume that the set-valued function \bar{m} is bounded in norm by M , i.e., that there exists a real number $M > 0$ such that

$$\forall (a, b) \in \mathcal{A} \times \mathcal{B}, \quad \sup_{d \in \bar{m}(a, b)} \|d\|_2 \leq M.$$

Then

$$\sup_{d \in \bar{m}(\mu)} \inf_{c \in \bar{m}(\nu)} \|d - c\|_2 \leq M \|\mu - \nu\|_1 \leq M \sqrt{N_{\mathcal{A}} N_{\mathcal{B}}} \|\mu - \nu\|_2,$$

where the norms in the right-hand side are respectively the ℓ^1 and ℓ^2 -norms between probability distributions.

Proof: Let d be an element of $\bar{m}(\mu)$; it can be written as

$$d = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \mu_{a, b} \theta_{a, b}$$

for some elements $\theta_{a, b} \in \bar{m}(a, b)$. We consider

$$c = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \nu_{a, b} \theta_{a, b},$$

which is an element of $\bar{m}(\nu)$. Then by the triangle inequality,

$$\|d - c\|_2 = \left\| \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} (\mu_{a, b} - \nu_{a, b}) \theta_{a, b} \right\|_2 \leq \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} |\mu_{a, b} - \nu_{a, b}| \|\theta_{a, b}\|_2 \leq M \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} |\mu_{a, b} - \nu_{a, b}|.$$

This entails the first claimed inequality. The second one follows from an application of the Cauchy-Schwarz inequality. \blacksquare

Necessary and sufficient condition for approachability. We state the condition in the theorem above, as well as the associated convergence rates. Efficient strategies can be deduced from the proof, which is based on Theorem 3.

Theorem 7 *Suppose that the set-valued function \bar{m} is bounded in norm by M . A closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$ is approachable (with pure or mixed actions) if and only if*

$$\forall \mathbf{y} \in \Delta(\mathcal{B}), \exists \mathbf{x} \in \Delta(\mathcal{A}), \quad \bar{m}(\mathbf{x}, \mathbf{y}) \subseteq \mathcal{C}.$$

In the latter case, the following convergence rates are achieved by an efficient strategy. With mixed actions taken and observed, for all strategies of the second player, with probability 1,

$$\sup_{d \in \bar{m}(\nu_T)} \inf_{c \in \mathcal{C}} \|c - d\|_2 \leq \frac{2M}{\sqrt{T}} \sqrt{N_{\mathcal{A}} N_{\mathcal{B}}}.$$

With pure actions taken and observed, for all $\delta \in (0, 1)$ and for all strategies of the second player, with probability at least $1 - \delta$,

$$\sup_{d \in \bar{m}(\pi_T)} \inf_{c \in \mathcal{C}} \|c - d\|_2 \leq \frac{2M}{\sqrt{T}} \sqrt{N_{\mathcal{A}} N_{\mathcal{B}}} \left(1 + 2\sqrt{\ln(2/\delta)}\right).$$

Proof: The condition is necessary. If the condition does not hold, then there exists $\mathbf{y}_0 \in \Delta(\mathcal{B})$ such that for every $\mathbf{x} \in \mathcal{A}$, the set $\bar{m}(\mathbf{x}, \mathbf{y}_0)$ is not included in \mathcal{C} , i.e., it contains at least one point not in \mathcal{C} . We then define a mapping $D : \Delta(\mathcal{A}) \rightarrow \mathbb{R}$ by

$$\forall \mathbf{x} \in \Delta(\mathcal{A}), \quad D(\mathbf{x}) = \sup_{d \in \bar{m}(\mathbf{x}, \mathbf{y}_0)} \inf_{c \in \mathcal{C}} \|c - d\|_2.$$

Since \mathcal{C} is closed, distances of given individual points to \mathcal{C} are achieved; therefore, by the choice of \mathbf{y}_0 , we get that $D(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Delta(\mathcal{A})$.

We now show that D is continuous on the compact set $\Delta(\mathcal{A})$; it thus attains its minimum, whose value we denote by $D_{\min} > 0$. More precisely, it suffices to show that for all $\mathbf{x}, \mathbf{x}' \in \Delta(\mathcal{A})$, the condition $\|\mathbf{x}' - \mathbf{x}\|_1 \leq \varepsilon$ implies that $D(\mathbf{x}) - D(\mathbf{x}') \leq M\varepsilon$. Indeed, fix $\delta > 0$ and let $d_{\delta, \mathbf{x}}$ be such that

$$D(\mathbf{x}) \leq \inf_{c \in \mathcal{C}} \|c - d_{\delta, \mathbf{x}}\|_2 + \delta. \quad (2)$$

By Lemma 6 (with the choices $\mu = \mathbf{x} \otimes \mathbf{y}_0$ and $\nu = \mathbf{x}' \otimes \mathbf{y}_0$) there exists $d_{\delta, \mathbf{x}'} \in \bar{m}(\mathbf{x}', \mathbf{y}_0)$ such that $\|d_{\delta, \mathbf{x}} - d_{\delta, \mathbf{x}'}\|_2 \leq M\varepsilon + \delta$. The triangle inequality entails that

$$\inf_{c \in \mathcal{C}} \|c - d_{\delta, \mathbf{x}}\|_2 \leq \inf_{c \in \mathcal{C}} \|c - d_{\delta, \mathbf{x}'}\|_2 + M\varepsilon + \delta.$$

Substituting in (2), we get that

$$D(\mathbf{x}) \leq M\varepsilon + 2\delta + \inf_{c \in \mathcal{C}} \|c - d_{\delta, \mathbf{x}'}\|_2 \leq M\varepsilon + 2\delta + D(\mathbf{x}'),$$

which, letting $\delta \rightarrow 0$, proves our continuity claim.

Assume now that the second player chooses at each round $\mathbf{y}_t = \mathbf{y}_0$ as his mixed action. In the case of mixed actions taken and observed, denoting by

$$\bar{\mathbf{x}}_T = \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t,$$

we get that $\nu_t = \bar{\mathbf{x}}_T \otimes \mathbf{y}_0$, and hence, for all strategies of the first player and for all $T \geq 1$,

$$\sup_{d \in \bar{m}(\nu_T)} \inf_{c \in \mathcal{C}} \|c - d\|_2 = D(\bar{\mathbf{x}}_T) \geq D_{\min} > 0,$$

which shows that \mathcal{C} is not approachable. The case of pure actions taken and observed is treated similarly, with the sole addition of a concentration argument. By repeated uses of the Hoeffding-Azuma inequality together with an application of the Borel-Cantelli lemma,

$$\delta_T = \|\pi_T - \nu_T\|_2 \longrightarrow 0 \quad \text{a.s.}$$

as $T \rightarrow \infty$. By applying Lemma 6 as above, we get

$$\sup_{d \in \bar{m}(\pi_T)} \inf_{c \in \mathcal{C}} \|c - d\|_2 \geq \sup_{d \in \bar{m}(\nu_T)} \inf_{c \in \mathcal{C}} \|c - d\|_2 - \delta_T \geq D_{\min} - \delta_T;$$

we simply take the \liminf in the above inequalities to conclude the argument. \blacksquare

Proof: The condition is sufficient. We first show that there exists a strategy of the first player such that, for all strategies of the opponent player, the sequences (π_T) or (ν_T) of the empirical distributions of actions converge to the set

$$\tilde{\mathcal{C}} = \{\mu \in \Delta(\mathcal{A} \times \mathcal{B}) : \bar{m}(\mu) \subseteq \mathcal{C}\}$$

in ℓ^2 -norm, at the rates prescribed by Theorem 3.

To do so, we identify probability distributions over $\mathcal{A} \times \mathcal{B}$ with vectors in $\mathbb{R}^{\mathcal{A} \times \mathcal{B}}$ and consider the vector-valued payoff function

$$m : (a, b) \in \mathcal{A} \times \mathcal{B} \mapsto \delta_{(a,b)} \in \mathbb{R}^{\mathcal{A} \times \mathcal{B}},$$

which we extend multi-linearly to $\Delta(\mathcal{A}) \times \Delta(\mathcal{B})$. We have that

$$\pi_T = \frac{1}{T} \sum_{t=1}^T m(A_t, B_t) \quad \text{and} \quad \nu_T = \frac{1}{T} \sum_{t=1}^T m(\mathbf{x}_t, \mathbf{y}_t)$$

and therefore only need to show that $\tilde{\mathcal{C}}$ is m -approachable (with pure or mixed actions).

Since \bar{m} is a linear function on $\Delta(\mathcal{A} \times \mathcal{B})$ and \mathcal{C} is convex, the set $\tilde{\mathcal{C}}$ is convex as well. In addition, since \mathcal{C} is closed, $\tilde{\mathcal{C}}$ is also closed. We can therefore apply the original version of the approachability theorem (Theorem 3). The desired existence result follows therefore from the fact that by assumption, for all $\mathbf{y} \in \Delta(\mathcal{B})$, there exists some $\mathbf{x} \in \Delta(\mathcal{A})$ such that $\mu = m(\mathbf{x}, \mathbf{y})$, the product-distribution between \mathbf{x} and \mathbf{y} , belongs to $\tilde{\mathcal{C}}$, as it satisfies $\bar{m}(\mu) = \bar{m}(\mathbf{x}, \mathbf{y}) \subseteq \mathcal{C}$.

Let $P_{\tilde{\mathcal{C}}}$ denote the projection operator onto $\tilde{\mathcal{C}}$. We therefore have proved the existence of efficient strategies such that, for all strategies of the second player, with probability $1 - \delta$,

$$\varepsilon_T := \left\| \pi_T - P_{\tilde{\mathcal{C}}}(\pi_T) \right\|_2 = \inf_{\mu \in \tilde{\mathcal{C}}} \|\pi_T - \mu\|_2 \leq \frac{2}{\sqrt{T}} \left(1 + \sqrt{2 \ln(2/\delta)}\right),$$

and with probability 1,

$$\varepsilon'_T := \left\| \nu_T - P_{\tilde{\mathcal{C}}}(\nu_T) \right\|_2 = \inf_{\mu \in \tilde{\mathcal{C}}} \|\nu_T - \mu\|_2 \leq \frac{2}{\sqrt{T}}.$$

Lemma 6 entails that the sets $\bar{m}(\pi_T)$ are included in $M\sqrt{N_{\mathcal{A}}N_{\mathcal{B}}}\varepsilon_T$ -neighborhoods of $\bar{m}(P_{\tilde{\mathcal{C}}}(\pi_T))$, and thus, by definition of $\tilde{\mathcal{C}}$, in $M\sqrt{N_{\mathcal{A}}N_{\mathcal{B}}}\varepsilon_T$ -neighborhoods of \mathcal{C} . A similar statement holds for the sets $\bar{m}(\nu_T)$ and this completes the proof. \blacksquare

4 Application to games with partial monitoring

A repeated vector-valued game with partial monitoring is described as follows. The players have respective finite action sets \mathcal{I} and \mathcal{J} . We denote by $r : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}^d$ the vector-valued payoff function of the first player and extend it multi-linearly to $\Delta(\mathcal{I}) \times \Delta(\mathcal{J})$. At each round, players simultaneously choose their actions $I_t \in \mathcal{I}$ and $J_t \in \mathcal{J}$, possibly at random according to probability distributions denoted by $\mathbf{p}_t \in \Delta(\mathcal{I})$ and $\mathbf{q}_t \in \Delta(\mathcal{J})$. At the end of a round, the first player does not observe J_t or $r(I_t, J_t)$ but only a signal. There is a finite set \mathcal{H} of possible signals; the feedback S_t that is given to the first player is drawn at random according to the distribution $H(I_t, J_t)$, where the mapping $H : \mathcal{I} \times \mathcal{J} \rightarrow \Delta(\mathcal{H})$ is known by the first player.

Example 1 Examples of such partial monitoring games are provided by, e.g., Cesa-Bianchi et al. (2006, Section 2), among which we can cite the apple tasting problem, the label-efficient prediction constraint, and the multi-armed bandit settings.

Some additional notation will be useful. We denote by R the norm of (the linear extension of) r ,

$$R = \max_{(i,j) \in \mathcal{I} \times \mathcal{J}} \|r(i, j)\|_2.$$

The cardinalities of the finite sets \mathcal{I} , \mathcal{J} , and \mathcal{H} will be referred to as $N_{\mathcal{I}}$, $N_{\mathcal{J}}$, and $N_{\mathcal{H}}$.

Definition 8 Let $\mathcal{C} \subseteq \mathbb{R}^d$ be some set; \mathcal{C} is r -approachable for the signaling structure H if there exists a strategy of the first player such that for all strategies of the second player,

$$\limsup_{T \rightarrow \infty} \inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right\|_2 = 0 \quad \text{a.s.}$$

That is, the first player has a strategy that ensures that the average of his vector-valued payoffs converges to the set \mathcal{C} , even if he only observes the random signals S_t as a feedback.

A necessary and sufficient condition for r -approachability with the signaling structure H was stated and proved by Perchet (2011); however, no efficient strategy was associated with it. To recall this condition we need some additional notation: for all $\mathbf{q} \in \Delta(\mathcal{J})$, we denote by $\tilde{H}(\mathbf{q})$ the element in $\Delta(\mathcal{H})^{\mathcal{I}}$ defined as follows. For all $i \in \mathcal{I}$, its i -th component is given by the following convex combination of probability distributions over \mathcal{H} ,

$$\tilde{H}(\mathbf{q})_i = H(i, \mathbf{q}) = \sum_{j \in \mathcal{J}} q_j H(i, j).$$

Finally, we denote by \mathcal{F} the set of feasible vectors of probability distributions over \mathcal{H} :

$$\mathcal{F} = \left\{ \tilde{H}(\mathbf{q}) : \mathbf{q} \in \Delta(\mathcal{J}) \right\}.$$

A generic element of \mathcal{F} will be denoted by $\sigma \in \mathcal{F}$. The necessary and sufficient condition of approachability of \mathcal{C} can now be stated.

Condition 1 *The signaling structure H , the vector-payoff function r , and the set \mathcal{C} satisfy*

$$\forall \mathbf{q} \in \Delta(\mathcal{J}), \exists \mathbf{p} \in \Delta(\mathcal{I}), \forall \mathbf{q}' \in \Delta(\mathcal{J}), \quad \tilde{H}(\mathbf{q}) = \tilde{H}(\mathbf{q}') \Rightarrow r(\mathbf{p}, \mathbf{q}') \in \mathcal{C}.$$

Defining the set-valued function \bar{m} , for all $\mathbf{p} \in \Delta(\mathcal{I})$ and $\sigma \in \mathcal{F}$, by

$$\bar{m}(\mathbf{p}, \sigma) = \left\{ r(\mathbf{p}, \mathbf{q}') : \mathbf{q}' \in \Delta(\mathcal{J}) \text{ such that } \tilde{H}(\mathbf{q}') = \sigma \right\},$$

the condition can be equivalently reformulated as

$$\forall \sigma \in \mathcal{F}, \exists \mathbf{p} \in \Delta(\mathcal{I}), \quad \bar{m}(\mathbf{p}, \sigma) \subseteq \mathcal{C}.$$

This condition is necessary. The next two sections show (in a constructive way and by providing efficient strategies) that this condition is sufficient for r -approachability of closed convex sets \mathcal{C} given the signaling structure H . We first prove that this condition is necessary.

When it is not satisfied, there exists a $\sigma_0 \in \mathcal{F}$ such that for all $\mathbf{p} \in \Delta(\mathcal{I})$, there exists a $\phi(\mathbf{p}) \in \Delta(\mathcal{J})$ such that $\tilde{H}(\phi(\mathbf{p})) = \sigma_0$ and $r(\mathbf{p}, \phi(\mathbf{p})) \notin \mathcal{C}$. We denote by $\tilde{H}^{-1}(\sigma_0)$ the set of all \mathbf{q} such that $\tilde{H}(\mathbf{q}) = \sigma_0$. We will consider a small subclass of the possible strategies of the second player: only those which prescribe him to play at each round the same element of $\tilde{H}^{-1}(\sigma_0)$. We will show that for all strategies of the first player, there exists a strategy of the second player of the form mentioned above such that, with some *positive* probability,

$$\limsup_{T \rightarrow \infty} \inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right\|_2 > 0. \quad (3)$$

We first note that by concentration of the measure (by the Hoeffding-Azuma inequality and the Borel-Cantelli lemma), if $\mathbf{q} \in \tilde{H}^{-1}(\sigma_0)$ is the element repeatedly played by the second player,

$$\lim_{T \rightarrow \infty} \left\| \frac{1}{T} \sum_{t=1}^T r(I_t, \mathbf{q}) - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right\|_2 = 0 \quad \text{a.s.} \quad (4)$$

Now, all considered strategies of the second player are indistinguishable to the first player, since they all induce the same vector σ_0 of probability distributions over signals. Therefore, the law of

$$\hat{\mathbf{p}}_T = \frac{1}{T} \sum_{t=1}^T \delta_{I_t}$$

only depends on T and σ_0 . We denote by $\bar{\mathbf{p}}_T$ the common expectation of the $\hat{\mathbf{p}}_T$ as the \mathbf{q} vary in $\tilde{H}^{-1}(\sigma_0)$; the expectation has to be understood with respect to the auxiliary randomizations taken (to draw the pure actions from the mixed actions \mathbf{p}_t and \mathbf{q} and to draw the signals).

We denote by $d_{\mathcal{C}}$ the distance to the closed convex set \mathcal{C} ; it is a continuous and convex function (see Boyd and Vandenberghe 2004, Example 3.16). In particular, it is bounded on the set of all feasible payoff vectors $r(\mathbf{p}, \mathbf{q})$, as \mathbf{p} and \mathbf{q} vary. By the dominated convergence theorem and in view of (4), to prove (3) it thus suffices to show that for all strategies of the first player, there exists a strategy of the second player in $\tilde{H}^{-1}(\sigma_0)$ such that

$$\limsup_{T \rightarrow \infty} \mathbb{E} \left[d_{\mathcal{C}} \left(r(\hat{\mathbf{p}}_T, \mathbf{q}) \right) \right] > 0.$$

By Jensen's inequality,

$$\mathbb{E} \left[d_{\mathcal{C}} \left(r(\widehat{\mathbf{p}}_T, \mathbf{q}) \right) \right] \geq d_{\mathcal{C}} \left(\mathbb{E} \left[r(\widehat{\mathbf{p}}_T, \mathbf{q}) \right] \right) = d_{\mathcal{C}} \left(r(\overline{\mathbf{p}}_T, \mathbf{q}) \right).$$

By the Bolzano-Weierstrass property, for all strategies of the first player, the sequence of the $\overline{\mathbf{p}}_T$ has values in the compact space $\Delta(\mathcal{I})$, so that it admits a converging subsequence, which we denote by $\overline{\mathbf{p}}_{\varphi(T)}$ and whose limit point we denote by $\overline{\mathbf{p}}_{\infty}$. (This limit point depends solely on the strategy of the first player and on σ_0 .) By considering $\mathbf{q} = \phi(\overline{\mathbf{p}}_{\infty})$ and putting the pieces together, we get that

$$\limsup_{T \rightarrow \infty} \mathbb{E} \left[d_{\mathcal{C}} \left(r(\widehat{\mathbf{p}}_T, \mathbf{q}) \right) \right] \geq \limsup_{T \rightarrow \infty} d_{\mathcal{C}} \left(r(\overline{\mathbf{p}}_{\varphi(T)}, \mathbf{q}) \right) = d_{\mathcal{C}} \left(r(\overline{\mathbf{p}}_{\infty}, \phi(\overline{\mathbf{p}}_{\infty})) \right) > 0,$$

since by definition of ϕ , the vector $r(\overline{\mathbf{p}}_{\infty}, \phi(\overline{\mathbf{p}}_{\infty}))$ is not in the closed convex set \mathcal{C} .

4.1 Approachability for deterministic feedback signals only depending on outcome

In this section, we assume that H is of the following form: it only contains Dirac masses, and these Dirac masses $H(i, j)$ only depend on j . Put differently, the signals S_t are deterministic functions of the actions J_t ; we thus denote by $h : \mathcal{J} \rightarrow \mathcal{H}$ the function such that $S_t = h(J_t)$ for all t and extend it linearly to $\Delta(\mathcal{J})$. The condition stated above takes the following simpler form (we assume with no loss of generality that all elements in \mathcal{H} are associated with at least one action $j \in \mathcal{J}$):

$$\forall \sigma \in \Delta(\mathcal{H}), \exists \mathbf{p} \in \Delta(\mathcal{I}), \quad \overline{m}(\mathbf{p}, \sigma) \subseteq \mathcal{C}, \quad (5)$$

where

$$\overline{m}(\mathbf{p}, \sigma) = \{ r(\mathbf{p}, \mathbf{q}') : \mathbf{q}' \in \Delta(\mathcal{J}) \text{ such that } h(\mathbf{q}') = \sigma \}.$$

It is straightforward that \overline{m} is linear, i.e., for all $\mathbf{p} \in \Delta(\mathcal{I})$ and $\sigma \in \Delta(\mathcal{H})$,

$$\overline{m}(\mathbf{p}, \sigma) = \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{H}} p_i \sigma_s \overline{m}(i, s).$$

In addition, \overline{m} is also bounded in norm by R .

Theorem 9 *A closed convex \mathcal{C} is r -approachable for the signaling structure h if and only if (5) holds. In this case, there exists an efficient strategy to do so, at the following rate: for all T , with probability at least $1 - \delta$,*

$$\inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right\|_2 \leq \frac{2R}{\sqrt{T}} \sqrt{N_{\mathcal{I}} N_{\mathcal{H}}} \left(1 + 2\sqrt{\ln(2/\delta)} \right).$$

Proof: We need only to show that the stated condition entails approachability (with an efficient strategy). Since by definition of \overline{m} and because of the particular signaling structure h ,

$$\frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \in \frac{1}{T} \sum_{t=1}^T \overline{m}(I_t, S_t),$$

it is enough to show that \mathcal{C} is \overline{m} -approachable (when the signals S_t are observed, which is the case). But Theorem 7 indicates that this is the case when (5) holds. \blacksquare

4.2 Approachability with general signaling structures

In this section we consider the case where the signal structure is general. We start from a technical lemma that is needed to show that $\overline{m}(\mathbf{p}, \sigma)$ can be written as a *finite* combination of sets of the form $\overline{m}(i, b)$. We then describe an efficient strategy for approachability followed by convergence rate guarantees.

4.2.1 A preliminary technical result.

With general signaling structures, \overline{m} is not linear, it only satisfies that for all $\mathbf{p} \in \Delta(\mathcal{I})$, all pairs $\sigma, \sigma' \in \mathcal{F}$, and all $\alpha \in [0, 1]$

$$\alpha \overline{m}(\mathbf{p}, \sigma) + (1 - \alpha) \overline{m}(\mathbf{p}, \sigma') \subseteq \overline{m}(\mathbf{p}, \alpha\sigma + (1 - \alpha)\sigma'),$$

with a strict inclusion in general. (Specific examples can be provided.) Therefore, a direct appeal to Theorem 7 is not possible anymore; however, a similar linearity property on a lifted finite set follows from the following geometric lemma whose proof can be found in Appendix A.1.

Parameters: an integer $L \geq 1$, an exploration parameter $\gamma \in [0, 1]$, an efficient strategy Ψ for \overline{m} -approachability of \mathcal{C}
Notation: $\mathbf{u} \in \Delta(\mathcal{I})$ is the uniform distribution over \mathcal{I} , $P_{\mathcal{F}}$ denotes the projection operator in ℓ^2 -norm of $\mathbb{R}^{\mathcal{H} \times \mathcal{I}}$ onto \mathcal{F}
Initialization: compute the finite set \mathcal{B} and the mapping $\Phi : \mathcal{F} \rightarrow \Delta(\mathcal{B})$ of Lemma 10, pick an arbitrary $\mathbf{x}_1 \in \Delta(\mathcal{I})$
For all blocks $n = 1, 2, \dots$,

1. define $\mathbf{p}_n = (1 - \gamma) \mathbf{x}_n + \gamma \mathbf{u}$;
2. for rounds $t = (n - 1)L + 1, \dots, nL$,
 - 2.1 drawn an action $I_t \in \mathcal{I}$ at random according to \mathbf{p}_n ;
 - 2.2 get the signal S_t ;
3. form the estimated vector of probability distributions over signals,

$$\tilde{\sigma}_n = \left(\frac{1}{L} \sum_{t=(n-1)L+1}^{nL} \frac{\mathbb{I}_{\{S_t=s\}} \mathbb{I}_{\{I_t=i\}}}{p_{I_t,n}} \right)_{(i,s) \in \mathcal{I} \times \mathcal{H}} ;$$

4. compute the projection $\hat{\sigma}_n = P_{\mathcal{F}}(\tilde{\sigma}_n)$;
 5. obtain $\mathbf{x}_{n+1} = \Psi(\mathbf{x}_1, \Phi(\hat{\sigma}_1), \dots, \mathbf{x}_n, \Phi(\hat{\sigma}_n))$.
-

Figure 1: The considered strategy, which plays in blocks.

Lemma 10 *There exist a finite subset $\mathcal{B} \subseteq \mathcal{F}$ and a mapping $\Phi : \mathcal{F} \rightarrow \Delta(\mathcal{B})$ such that*

$$\forall \sigma \in \mathcal{F}, \forall \mathbf{p} \in \Delta(\mathcal{I}), \quad \overline{m}(\mathbf{p}, \sigma) = \sum_{i \in \mathcal{I}} \sum_{b \in \mathcal{B}} p_i \Phi_b(\sigma) \overline{m}(i, b),$$

where we denoted the convex weight vector $\Phi(\sigma) \in \Delta(\mathcal{B})$ by $(\Phi_b(\sigma))_{b \in \mathcal{B}}$.

Definition 11 *We denote by $\overline{\overline{m}}$ the linear extension to $\Delta(\mathcal{I} \times \mathcal{B})$ of the restriction of \overline{m} to $\mathcal{I} \times \mathcal{B}$, so that for all $\mathbf{p} \in \Delta(\mathcal{I})$ and $\sigma \in \mathcal{F}$,*

$$\overline{\overline{m}}(\mathbf{p}, \sigma) = \overline{\overline{m}}(\mathbf{p}, \Phi(\sigma)).$$

Remark 1 The proof shows that Φ is piecewise linear on a finite decomposition of \mathcal{F} ; it is therefore Lipschitz on \mathcal{F} . We denote by κ_{Φ} its Lipschitz constant with respect to the ℓ^2 -norm.

4.2.2 An efficient strategy.

The approaching strategy for the original problem is based on an efficient strategy Ψ for \overline{m} -approachability of \mathcal{C} , provided by Theorem 7 and thus solving repeatedly minimax problems of the form (1). We therefore first prove the existence of such a Ψ ; to do so, we show that the condition in Theorem 7 is satisfied, that is, that for all $\mathbf{y} \in \Delta(\mathcal{B})$, there exists a $\mathbf{p} \in \Delta(\mathcal{I})$ such that $\overline{\overline{m}}(\mathbf{p}, \mathbf{y}) \subseteq \mathcal{C}$. By linearity of $\overline{\overline{m}}$ (for the following equality) and by definition of \overline{m} (for the following inclusion),

$$\overline{\overline{m}}(\mathbf{p}, \mathbf{y}) = \sum_{b \in \mathcal{B}} y_b \overline{m}(\mathbf{p}, b) \subseteq \overline{\overline{m}}\left(\mathbf{p}, \sum_{b \in \mathcal{B}} y_b b\right).$$

The existence of the desired \mathbf{p} is therefore ensured by Condition 1, applied with $\sigma = \sum_{b \in \mathcal{B}} y_b b$.

We consider the strategy described in Figure 1. It forces exploration at a γ rate, as is usual in situations with partial monitoring. One of its key ingredient, that conditionally unbiased estimators are available, is extracted from Lugosi et al. (2008, Section 6): we consider

$$\hat{H}_t = \frac{\mathbb{I}_{\{S_t=s\}} \mathbb{I}_{\{I_t=i\}}}{p_{I_t,n}} \in \mathbb{R}^{\mathcal{H} \times \mathcal{I}};$$

averaging over the respective random draws of I_t and S_t according to \mathbf{p}_n and $H(I_t, J_t)$, i.e., taking its conditional expectation \mathbb{E}_t with respect to \mathbf{p}_n and J_t , we get

$$\mathbb{E}_t[\hat{H}_t] = \tilde{H}(\delta_{J_t}). \quad (6)$$

This is why, by concentration-of-the-measure argument, we will be able to show that for L large enough, $\tilde{\sigma}_n$ is close to $\tilde{H}(\hat{\mathbf{q}}_n)$, where

$$\hat{\mathbf{q}}_n = \frac{1}{L} \sum_{t=(n-1)L+1}^{nL} \delta_{J_t}. \quad (7)$$

Actually, since $\mathcal{F} \subseteq \Delta(\mathcal{H})^{\mathcal{I}}$, we have a natural embedding of \mathcal{F} into $\mathbb{R}^{\mathcal{H} \times \mathcal{I}}$ and we can define $P_{\mathcal{F}}$, the convex projection operator onto \mathcal{F} (in ℓ^2 -norm). Instead of using directly $\tilde{\sigma}_n$, we consider in our strategy $\hat{\sigma}_n = P_{\mathcal{F}}(\tilde{\sigma}_n)$, which is even closer to $\tilde{H}(\hat{\mathbf{q}}_n)$.

4.2.3 Performance guarantee.

We provide a performance bound for fixed parameters γ and L . Adaptation to $T \rightarrow \infty$ can be performed by resorting to a standard doubling trick (see, e.g., Cesa-Bianchi and Lugosi 2006, page 17). The proof is provided in Appendix A.2.

Theorem 12 *Consider the strategy in Figure 1 with parameters $\gamma \in [0, 1]$ and $L \geq 1$ and fed with an efficient strategy Ψ for \overline{m} -approachability of \mathcal{C} , provided by Theorem 7. Then, for all round $T \geq L + 1$ and with probability at least $1 - \delta$,*

$$\begin{aligned} \inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right\|_2 &\leq \frac{2L}{T} R + 4R \sqrt{\frac{\ln((2T)/(L\delta))}{T}} + 2\gamma R + \frac{2R}{\sqrt{T/L-1}} \sqrt{N_{\mathcal{I}} N_{\mathcal{B}}} \\ &\quad + R\kappa_{\Phi} N_{\mathcal{H}} \sqrt{N_{\mathcal{I}}} \left(\sqrt{\frac{2N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}} T}{L\delta}} + \frac{1}{3} \frac{N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}} T}{L\delta} \right). \end{aligned}$$

In particular, for all $T \geq 1$, the choices of $L = \lceil T^{3/5} \rceil$ and $\gamma = T^{-1/5}$ imply that with probability at least $1 - \delta$,

$$\inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right\|_2 \leq \square \left(T^{-1/5} \sqrt{\ln \frac{T}{\delta}} + T^{-2/5} \ln \frac{T}{\delta} \right)$$

for some constant \square depending only on \mathcal{C} and on the game (r, H) at hand.

5 Application to regret minimization

In this section we analyze external and internal regret minimization in repeated games with partial monitoring from the approachability perspective. Using the results developed for vector-valued games with partial monitoring, we show how to efficiently minimize regret in both setups.

5.1 External regret

We consider in this section the framework and aim introduced by Rustichini (1999) and studied, sometimes in special cases, by Piccolboni and Schindelhauer (2001), Mannor and Shimkin (2003), Cesa-Bianchi et al. (2006), Lugosi et al. (2008).

Scalar payoffs are obtained (but not observed) by the first player: the payoff function r is a mapping $\mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}$; we still denote by R a bound on $|r|$. We define in this section

$$\hat{\mathbf{q}}_T = \frac{1}{T} \sum_{t=1}^T \delta_{J_t}$$

as the empirical distribution of the actions taken by the second player. The external regret of the first player at round T equals by definition

$$R_T^{\text{ext}} = \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\hat{\mathbf{q}}_T)) - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t),$$

where $\rho : \Delta(\mathcal{I}) \times \mathcal{F}$ is defined as follows: for all $\mathbf{p} \in \Delta(\mathcal{I})$ and $\sigma \in \mathcal{F}$,

$$\rho(\mathbf{p}, \sigma) = \min \left\{ r(\mathbf{p}, \mathbf{q}) : \mathbf{q} \text{ such that } \tilde{H}(\mathbf{q}) = \sigma \right\}.$$

The function ρ is continuous in its first argument and therefore the supremum in the defining expression of R_T^{ext} is a maximum.

We recall briefly why, intuitively, this is the natural notion of external regret to consider in this case. Indeed, the first term in the definition of R_T^{ext} is (close to) the worst-case average payoff obtained by the first player when playing consistently a mixed action \mathbf{p} against a sequence of mixed actions inducing the laws on the signals.

The following result is an easy consequence of Theorem 12, as is explained below; it corresponds to the main result of Lugosi et al. (2008), with the same convergence rate but with a different strategy.

Corollary 13 *For all T , the first player has an efficient strategy such that, for all strategies of the second player and with probability at least $1 - \delta$,*

$$R_T^{\text{ext}} \leq \square \left(T^{-1/5} \sqrt{\ln \frac{T}{\delta}} + T^{-2/5} \ln \frac{T}{\delta} \right)$$

for some constant \square depending only on the game (r, H) at hand.

The proof below is actually an extension to the setting of partial monitoring of the original proof and strategy of Blackwell (1956b) for the case of external regret under full monitoring: the vector-payoff function \underline{r} and the set \mathcal{C} considered in our proof are equal to the ones considered by Blackwell in the case of full monitoring.

Proof: As usual, we embed $\Delta(\mathcal{J})$ into $\mathbb{R}^{\mathcal{J}}$ so that in this proof we will be working in the vector space $\mathbb{R} \times \mathbb{R}^{\mathcal{J}}$. We consider the convex set \mathcal{C} and the vector-valued payoff function \underline{r} respectively defined by

$$\mathcal{C} = \left\{ (z, \mathbf{q}) \in \mathbb{R} \times \Delta(\mathcal{J}) : z \geq \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\mathbf{q})) \right\} \quad \text{and} \quad \underline{r}(i, j) = \begin{bmatrix} r(i, j) \\ \delta_j \end{bmatrix},$$

for all $(i, j) \in \mathcal{I} \times \mathcal{J}$. By Theorem 12, to show that \mathcal{C} is \underline{r} -approachable, we associate with each $\mathbf{q} \in \Delta(\mathcal{J})$ an element $\phi(\mathbf{q}) \in \Delta(\mathcal{I})$ such that

$$\phi(\mathbf{q}) \in \operatorname{argmax}_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\mathbf{q})).$$

Then, given any $\mathbf{q} \in \Delta(\mathcal{J})$, we note that for all \mathbf{q}' satisfying $\tilde{H}(\mathbf{q}') = \tilde{H}(\mathbf{q})$, we have, by definition of ρ ,

$$r(\phi(\mathbf{q}), \mathbf{q}') \geq \rho(\phi(\mathbf{q}), \tilde{H}(\mathbf{q})) = \rho(\phi(\mathbf{q}), \tilde{H}(\mathbf{q}')),$$

which shows that $\underline{r}(\phi(\mathbf{q}), \mathbf{q}') \in \mathcal{C}$. The required condition is thus satisfied.

To exhibit the convergence rates, we use the fact that the mapping

$$\mathbf{q} \in \Delta(\mathcal{J}) \mapsto \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\mathbf{q}))$$

is Lipschitz, with Lipschitz constant in ℓ^2 -norm denoted by L_ρ ; this fact is proved below. Now, the regret is non positive as soon as $\sum_{t=1}^T \underline{r}(I_t, J_t)/T$ belongs to \mathcal{C} ; we therefore only need to consider the case when this average is not in \mathcal{C} . In the latter case, we denote by $(\tilde{r}_T, \tilde{\mathbf{q}}_T)$ its projection in ℓ^2 -norm onto \mathcal{C} . We have first that the defining inequality of \mathcal{C} is an equality on its border, so that

$$\tilde{r}_T = \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\tilde{\mathbf{q}}_T));$$

and second, that

$$\begin{aligned} R_T^{\text{ext}} &= \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\hat{\mathbf{q}}_T)) - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \\ &\leq \left| \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\hat{\mathbf{q}}_T)) - \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \tilde{H}(\tilde{\mathbf{q}}_T)) \right| + \left| \tilde{r}_T - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right| \\ &\leq L_\rho \|\hat{\mathbf{q}}_T - \tilde{\mathbf{q}}_T\|_2 + \left| \tilde{r}_T - \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \right| \\ &\leq (L_\rho + 1) \left\| \begin{bmatrix} \tilde{r}_T \\ \tilde{\mathbf{q}}_T \end{bmatrix} - \frac{1}{T} \sum_{t=1}^T \underline{r}(I_t, J_t) \right\|_2 \\ &= (L_\rho + 1) \inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T \underline{r}(I_t, J_t) \right\|_2. \end{aligned}$$

The rates follow from the ones indicated in Theorem 12.

It only remains to prove the indicated Lipschitzness. We prove that $L_\rho = R\kappa_\Phi\sqrt{N_{\mathcal{I}}N_{\mathcal{J}}N_{\mathcal{B}}}$. On the one hand, for every $i \in \mathcal{I}$, the mappings $\mathbf{q} \in \Delta(\mathcal{J}) \mapsto H(i, \mathbf{q})$ are $\sqrt{N_{\mathcal{J}}}$ -Lipschitz as the $\|H(i, j)\|_2$ are bounded by 1 for all j . Thus, the mapping $\mathbf{q} \in \Delta(\mathcal{J}) \mapsto \tilde{H}(\mathbf{q})$ is $\sqrt{N_{\mathcal{I}}N_{\mathcal{J}}}$ -Lipschitz. On the other hand, we have by definition that for all $\mathbf{p} \in \Delta(\mathcal{I})$ and $\sigma \in \mathcal{F}$,

$$\rho(\mathbf{p}, \sigma) = \min \bar{m}(\mathbf{p}, \Phi(\sigma)),$$

and that (by Remark 1) the mapping $\sigma \in \mathcal{F} \mapsto \Phi(\sigma)$ is κ_Φ -Lipschitz; this entails, by Lemma 6, that for all $\mathbf{p} \in \Delta(\mathcal{I})$, the mapping $\sigma \in \mathcal{F} \mapsto \rho(\mathbf{p}, \sigma)$ is $R\sqrt{N_{\mathcal{B}}}\kappa_\Phi$ -Lipschitz. In particular, since the latter Lipschitz constant is independent of \mathbf{p} , the mapping

$$\sigma \in \mathcal{F} \mapsto \max_{\mathbf{p} \in \Delta(\mathcal{I})} \rho(\mathbf{p}, \sigma)$$

is $R\sqrt{N_{\mathcal{B}}}\kappa_\Phi$ -Lipschitz as well. Combining the two Lipschitz mappings yields yet another Lipschitz mapping, whose Lipschitz constant is the product of the Lipschitz constants of the base two mappings. ■

5.2 Internal / swap regret

Foster and Vohra (1999) defined internal regret with full monitoring as follows. A player has no internal regret if, for every action $i \in \mathcal{I}$, he has no external regret on the stages when this specific action i was played. In other words, i is the best response to the empirical distribution of action of the other player on these stages.

With partial monitoring, the first player evaluates his payoffs in some pessimistic way through the function ρ defined above. This function is not linear over $\Delta(\mathcal{I})$ in general (it is concave), so that the best responses are not necessarily pure actions $i \in \mathcal{I}$ but mixed actions, i.e., elements of $\Delta(\mathcal{I})$. Following Lehrer and Solan (2007) we therefore should partition the stages not depending on the pure actions actually played but on the mixed actions $\mathbf{p}_t \in \Delta(\mathcal{I})$ used to draw them. To this end, it is convenient to assume that the strategies of the first player need to pick these mixed actions in a finite (but possibly thin) grid of $\Delta(\mathcal{I})$, which we denote by $\{\mathbf{p}_g, g \in \mathcal{G}\}$, where \mathcal{G} is a finite set. At each round, the first player picks an index $G_t \in \mathcal{G}$ and uses the distribution \mathbf{p}_{G_t} to draw his action I_t . Up to a standard concentration-of-the-measure argument, we will measure the payoff at round t with $r(\mathbf{p}_{G_t}, J_t)$ rather than with $r(I_t, J_t)$.

For each $g \in \mathcal{G}$, we denote by $N_T(g)$ the number of stages in $\{1, \dots, T\}$ for which we had $G_t = g$ and, whenever $N_T(g) > 0$,

$$\hat{\mathbf{q}}_{T,g} = \frac{1}{N_T(g)} \sum_{t:G_t=g} \delta_{J_t}.$$

We define $\hat{\mathbf{q}}^{T,g}$ is an arbitrary way when $N_T(g) = 0$. The internal regret of the first player at round T is measured as

$$R_T^{\text{int}} = \max_{g,g' \in \mathcal{G}} \frac{N_T(g)}{T} \left(\rho(\mathbf{p}_{g'}, \tilde{H}(\hat{\mathbf{q}}_{T,g})) - r(\mathbf{p}_g, \hat{\mathbf{q}}_{T,g}) \right).$$

Actually, our proof technique rather leads to the minimization of some swap regret; see Blum and Mansour (2007) for the definition of swap regret in full monitoring.

$$R_T^{\text{swap}} = \sum_{g \in \mathcal{G}} \frac{N_T(g)}{T} \left(\max_{g' \in \mathcal{G}} \rho(\mathbf{p}_{g'}, \tilde{H}(\hat{\mathbf{q}}_{T,g})) - r(\mathbf{p}_g, \hat{\mathbf{q}}_{T,g}) \right).$$

Again, the following bound on the swap regret easily follows from Theorem 12; it provides an efficient strategy to control the swap regret, thus also the internal regret. It therefore improves on the results of Lehrer and Solan (2007), Perchet (2009), two articles which presented inefficient strategies to do so (strategies based on auxiliary strategies using a grid that needs to be refined over time and whose complexities is exponential in the size of these grids). Moreover, we provide convergence rates.

Corollary 14 *For all T , the first player has an efficient strategy such that, for all strategies of the second player and with probability at least $1 - \delta$,*

$$R_T^{\text{swap}} \leq \square \left(T^{-1/5} \sqrt{\ln \frac{T}{\delta}} + T^{-2/5} \ln \frac{T}{\delta} \right)$$

for some constant \square depending only on the game (r, H) at hand and on the size of the finite grid \mathcal{G} .

The proof of this corollary is based on ideas similar to the ones used in the proof of Corollary 13; because of the page limit, it is deferred to Appendix A.3.

References

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A COLT appendix beyond the page limit

A.1 Proof of Lemma 10

Proof: Rambau and Ziegler (1996, Proposition 2.4) state that since \tilde{H} is linear on the polytope $\Delta(\mathcal{J})$, its inverse application \tilde{H}^{-1} is a piecewise linear mapping of \mathcal{F} into the subsets of $\Delta(\mathcal{J})$, which means that there exists a finite decomposition of \mathcal{F} into polytopes $\{P_1, \dots, P_K\}$ each on which \tilde{H}^{-1} is linear. Up to a triangulation (see, e.g. Goodman and O'Rourke 2004, Chapter 14), we can assume that each P_k is a simplex. Denote by $\mathcal{B}_k \subseteq \mathcal{F}$ the set of vertices of P_k ; then, the finite subset stated in the lemma is

$$\mathcal{B} = \bigcup_{k=1}^K \mathcal{B}_k,$$

the set of all vertices of all the simplices.

Fix any $\sigma \in \mathcal{F}$. It belongs to some simplex P_k , so that there exists a convex decomposition $\sigma = \sum_{b \in \mathcal{B}_k} \lambda_b b$; this decomposition is unique within the simplex P_k . If σ belongs to two different simplices, then it actually belongs to their common face and the two decompositions coincide (some coefficients λ_b in the above decomposition are null). All in all, with each $\sigma \in \mathcal{F}$, we could associate a unique decomposition in \mathcal{B} ,

$$\sigma = \sum_{b \in \mathcal{B}} \Phi_b(\sigma) b,$$

where the coefficients $(\Phi_b(\sigma))_{b \in \mathcal{B}}$ form a convex weight vector over \mathcal{B} , i.e., belong to $\Delta(\mathcal{B})$; in addition, $\Phi_b(\sigma) > 0$ only if $b \in \mathcal{B}_k$ and $\sigma \in P_k$.

Since \tilde{H}^{-1} is linear on each simplex P_1, \dots, P_K , we therefore get

$$\tilde{H}^{-1}(\sigma) = \sum_{b \in \mathcal{B}} \Phi_b(\sigma) \tilde{H}^{-1}(b).$$

Finally, the result is a consequence of the fact that

$$\bar{m}(\mathbf{p}, \sigma) = r\left(\mathbf{p}, \tilde{H}^{-1}(\sigma)\right) = r\left(\mathbf{p}, \sum_{b \in \mathcal{B}} \Phi_b(\sigma) \tilde{H}^{-1}(b)\right),$$

which implies, by linearity of r , that

$$\bar{m}(\mathbf{p}, \sigma) = \sum_{b \in \mathcal{B}} \Phi_b(\sigma) r\left(\mathbf{p}, \tilde{H}^{-1}(b)\right) = \sum_{b \in \mathcal{B}} \Phi_b(\sigma) \bar{m}(\mathbf{p}, b).$$

The proof is concluded by noting that by definition, for all $\sigma \in \mathcal{F}$, the applications $\mathbf{p} \in \Delta(\mathcal{I}) \mapsto \bar{m}(\mathbf{p}, \sigma)$ are linear. \blacksquare

A.2 Proof of Theorem 12

Proof: We write T as $T = NL + k$ where N is an integer and $0 \leq k \leq L - 1$ and will show successively that (possibly with overwhelming probability only) the following statements hold.

$$\frac{1}{T} \sum_{t=1}^T r(I_t, J_t) \quad \text{is close to} \quad \frac{1}{NL} \sum_{t=1}^{NL} r(I_t, J_t); \quad (8)$$

$$\frac{1}{NL} \sum_{t=1}^{NL} r(I_t, J_t) \quad \text{is close to} \quad \frac{1}{N} \sum_{n=1}^N r(\mathbf{p}_n, \hat{\mathbf{q}}_n); \quad (9)$$

$$\frac{1}{N} \sum_{n=1}^N r(\mathbf{p}_n, \hat{\mathbf{q}}_n) \quad \text{is close to} \quad \frac{1}{N} \sum_{n=1}^N r(\mathbf{x}_n, \hat{\mathbf{q}}_n); \quad (10)$$

$$\frac{1}{N} \sum_{n=1}^N r(\mathbf{x}_n, \hat{\mathbf{q}}_n) \quad \text{belongs to the set} \quad \frac{1}{N} \sum_{n=1}^N \bar{m}(\mathbf{x}_n, \tilde{H}(\hat{\mathbf{q}}_n));$$

$$\frac{1}{N} \sum_{n=1}^N \bar{m}(\mathbf{x}_n, \tilde{H}(\hat{\mathbf{q}}_n)) \quad \text{is equal to the set} \quad \frac{1}{N} \sum_{n=1}^N \bar{m}(\mathbf{x}_n, \Phi(\tilde{H}(\hat{\mathbf{q}}_n)));$$

$$\frac{1}{N} \sum_{n=1}^N \overline{m} \left(\mathbf{x}_n, \Phi \left(\tilde{H}(\hat{\mathbf{q}}_n) \right) \right) \quad \text{is close to the set} \quad \frac{1}{N} \sum_{n=1}^N \overline{m} \left(\mathbf{x}_n, \Phi(\hat{\sigma}_n) \right); \quad (11)$$

$$\frac{1}{N} \sum_{n=1}^N \overline{m} \left(\mathbf{x}_n, \Phi(\hat{\sigma}_n) \right) \quad \text{is close to the set} \quad \mathcal{C}; \quad (12)$$

where we recall that the notation $\hat{\mathbf{q}}_n$ was defined in (7). Actually, we will show below the numbered statements only; the first unnumbered statement is immediate by the very definition of \overline{m} and the second one follows from Definition 11.

Step 1: the term (8). A direct calculation decomposing the sum over T elements into a sum over the NL first elements and the k remaining ones shows that

$$\left\| \frac{1}{T} \sum_{t=1}^T r(I_t, J_t) - \frac{1}{NL} \sum_{t=1}^{NL} r(I_t, J_t) \right\|_2 \leq R \left(\frac{k}{T} + \left(\frac{1}{NL} - \frac{1}{T} \right) NL \right) \leq \frac{2k}{T} R \leq \frac{2L}{T} R.$$

Step 2: the term (9). We note that by defining \mathbb{E}_t the conditional expectation with respect to $(I_1, S_1, J_1), \dots, (I_{t-1}, S_{t-1}, J_{t-1})$ and J_t , which fixes the values of the law \mathbf{p}'_t of I_t and the value of J_t , we have

$$\mathbb{E}_t[r(I_t, J_t)] = r(\mathbf{p}'_t, J_t).$$

We note that by definition of the forecaster, $\mathbf{p}'_t = \mathbf{p}_n$ if t belongs to the n -th block. By a version of the Hoeffding-Azuma inequality for sums of Hilbert space-valued martingale differences proved in³ Chen and White (1996, Lemma 3.2), we therefore get that with probability at least $1 - \delta$,

$$\left\| \frac{1}{NL} \sum_{t=1}^{NL} r(I_t, J_t) - \frac{1}{N} \sum_{n=1}^N r(\mathbf{p}_n, \hat{\mathbf{q}}_n) \right\|_2 \leq 4R \sqrt{\frac{\ln(2/\delta)}{T}}.$$

Step 3: the term (10). Since by definition $\mathbf{p}_n = (1 - \gamma) \mathbf{x}_n + \gamma \mathbf{u}$, we get

$$\left\| \frac{1}{N} \sum_{n=1}^N r(\mathbf{p}_n, \hat{\mathbf{q}}_n) - \frac{1}{N} \sum_{n=1}^N r(\mathbf{x}_n, \hat{\mathbf{q}}_n) \right\|_2 \leq 2\gamma R.$$

Step 4: the term (11). We fix a given block n . It can be extracted from Lugosi et al. (2008, proof of Theorem 6.1) that with probability $1 - \delta$,

$$\left\| \hat{\sigma}_n - \tilde{H}(\hat{\mathbf{q}}_n) \right\|_2 \leq \sqrt{N_{\mathcal{I}} N_{\mathcal{H}}} \left(\sqrt{\frac{2N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}}}{\delta}} + \frac{1}{3} \frac{N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}}}{\delta} \right). \quad (13)$$

(For the convenience of the reviewers this extracted statement is however proved again in Appendix B.2 below.) Since Φ is Lipschitz (see Remark 1), with Lipschitz constant in ℓ^2 -norm denoted by κ_{Φ} , we get that with probability $1 - \delta$,

$$\left\| \Phi(\hat{\sigma}_n) - \Phi \left(\tilde{H}(\hat{\mathbf{q}}_n) \right) \right\|_2 \leq \kappa_{\Phi} \sqrt{N_{\mathcal{I}} N_{\mathcal{H}}} \left(\sqrt{\frac{2N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}}}{\delta}} + \frac{1}{3} \frac{N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}}}{\delta} \right).$$

By a union bound, the above bound holds for all blocks $n = 1, \dots, N$ with probability at least $1 - N\delta$. Finally, an application of Lemma 6 shows that

$$\frac{1}{N} \sum_{n=1}^N \overline{m} \left(\mathbf{x}_n, \Phi \left(\tilde{H}(\hat{\mathbf{q}}_n) \right) \right) \quad \text{is in a } \varepsilon_T\text{-neighborhood (in } \ell^2\text{-norm) of} \quad \frac{1}{N} \sum_{n=1}^N \overline{m} \left(\mathbf{x}_n, \Phi(\hat{\sigma}_n) \right),$$

where

$$\varepsilon_T = R \sqrt{N_{\mathcal{H}}} \times \kappa_{\Phi} \sqrt{N_{\mathcal{I}} N_{\mathcal{H}}} \left(\sqrt{\frac{2N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}}}{\delta}} + \frac{1}{3} \frac{N_{\mathcal{I}}}{\gamma L} \ln \frac{2N_{\mathcal{I}} N_{\mathcal{H}}}{\delta} \right).$$

Step 5: the term (12). Since \mathcal{C} is \overline{m} -approachable and by definition of the choices of the \mathbf{x}_n in Figure 1, we get by Theorem 7, with probability 1,

$$\inf_{c \in \mathcal{C}} \left\| c - \frac{1}{N} \sum_{n=1}^N \overline{m} \left(\mathbf{x}_n, \Phi(\hat{\sigma}_n) \right) \right\|_2 \leq \frac{2R}{\sqrt{N}} \sqrt{N_{\mathcal{I}} N_{\mathcal{B}}} \leq \frac{2R}{\sqrt{T/L - 1}} \sqrt{N_{\mathcal{I}} N_{\mathcal{B}}},$$

since $T/L \leq N + k/L \leq N + 1$.

The proof is concluded by putting the pieces together thanks to a triangle inequality, by noting that $T/L \leq N + 1$, and by considering $L\delta/T \leq \delta/(N + 1)$ instead of δ . \blacksquare

³We use the fact that $\sqrt{u} e^{-u} \leq e^{-u/2}$ for all $u \geq 0$.

A.3 Proof of Corollary 14

We provide here the existence proof of efficient strategies minimizing the swap regret. The proof follows the same lines as the one of Corollary 13.

Proof: In this proof we will be working in the vector space $(\mathbb{R} \times \mathbb{R}^{\mathcal{J}})^{\mathcal{G}}$. We first extend linearly r from $\Delta(\mathcal{I}) \times \Delta(\mathcal{J})$ to $\Delta(\mathcal{I}) \times \mathbb{R}_+^{\mathcal{J}}$ and extend also continuously (but not linearly) $\rho : \mathcal{G} \times \Delta(\mathcal{J}) \rightarrow \mathbb{R}$ into a mapping $\zeta : \mathcal{G} \times \mathbb{R}_+^{\mathcal{J}} \rightarrow \mathbb{R}$ as follows: for all $g \in \mathcal{G}$ and $\mathbf{v} \in \mathbb{R}_+^{\mathcal{J}}$,

$$\zeta(\mathbf{p}_g, \mathbf{v}) = \begin{cases} 0 & \text{if } \|\mathbf{v}\|_1 = 0, \\ \|\mathbf{v}\|_1 \rho\left(\mathbf{p}_g, \tilde{H}\left(\frac{\mathbf{v}}{\|\mathbf{v}\|_1}\right)\right) & \text{if } \|\mathbf{v}\|_1 > 0. \end{cases}$$

The convex set \mathcal{C} and the vector-valued payoff function \underline{r} are then respectively defined by

$$\mathcal{C} = \left\{ (z_g, \mathbf{v}_g) \in (\mathbb{R} \times \mathbb{R}_+^{\mathcal{J}})^{\mathcal{G}} : \forall g \in \mathcal{G}, z_g \geq \max_{g' \in \mathcal{G}} \zeta(\mathbf{p}_{g'}, \mathbf{v}_g) \right\}$$

and, for all $(g, j) \in \mathcal{G} \times \mathcal{J}$,

$$\underline{r}(g, j) = \left[\begin{array}{c} r(\mathbf{p}_g, j) \mathbb{I}_{\{g'=g\}} \\ \delta_j \mathbb{I}_{\{g'=g\}} \end{array} \right]_{g' \in \mathcal{G}}.$$

To show that \mathcal{C} is \underline{r} -approachable, we associate with each $\mathbf{q} \in \Delta(\mathcal{J})$ an element $g^*(\mathbf{q}) \in \mathcal{G}$ such that

$$g^*(\mathbf{q}) \in \operatorname{argmax}_{g \in \mathcal{G}} \rho(\mathbf{p}_g, \tilde{H}(\mathbf{q})).$$

Then, given any $\mathbf{q} \in \Delta(\mathcal{J})$, we note that for all \mathbf{q}' satisfying $\tilde{H}(\mathbf{q}') = \tilde{H}(\mathbf{q})$, the components of the vector $\underline{r}(g^*(\mathbf{q}), \mathbf{q}')$ are all null but the ones corresponding to $g^*(\mathbf{q})$, for which we have

$$\max_{g' \in \mathcal{G}} \zeta(\mathbf{p}_{g'}, \mathbf{q}') = \max_{g' \in \mathcal{G}} \rho(\mathbf{p}_{g'}, \tilde{H}(\mathbf{q}')) = \max_{g' \in \mathcal{G}} \rho(\mathbf{p}_{g'}, \tilde{H}(\mathbf{q})) = \rho(\mathbf{p}_{g^*(\mathbf{q})}, \tilde{H}(\mathbf{q})) \leq r(\mathbf{p}_{g^*(\mathbf{q})}, \mathbf{q}'),$$

where the last line is by definition of ρ . Therefore, $\underline{r}(g^*(\mathbf{q}), \mathbf{q}') \in \mathcal{C}$. The required condition in Theorem 12 is thus satisfied, hence the desired approachability.

We now exhibit the convergence rates. As in the proof of Corollary 13, we need only to consider the case where $\sum_{t=1}^T \underline{r}(I_t, J_t)/T$ is not in \mathcal{C} , for otherwise, the swap regret is non positive. We denote by $(\tilde{r}_{T,g}, \tilde{\mathbf{v}}_{T,g})_{g \in \mathcal{G}}$ the projection in ℓ^2 -norm of $\sum_{t=1}^T \underline{r}(I_t, J_t)/T$ onto \mathcal{C} , and by $\hat{\mathbf{v}}_{T,g} = (N_T(g)/T) \hat{\mathbf{q}}_{T,g}$ the realized frequency of playing each $g \in \mathcal{G}$. Since the projection lies on the border of \mathcal{C} , we have that for all $g \in \mathcal{G}$,

$$\tilde{r}_{T,g} = \max_{g' \in \mathcal{G}} \zeta(\mathbf{p}_{g'}, \tilde{\mathbf{v}}_{T,g}).$$

We will prove below that

$$\mathbf{v} \in \mathbb{R}_+^{\mathcal{J}} \longmapsto \max_{g \in \mathcal{G}} \zeta(\mathbf{p}_g, \mathbf{v})$$

is L_ζ -Lipschitz, for some constant $L_\zeta > 0$. Then, as for the external regret,

$$\begin{aligned} R_T^{\text{swap}} &= \sum_{g \in \mathcal{G}} \frac{N_T(g)}{T} \left(\max_{g' \in \mathcal{G}} \rho(\mathbf{p}_{g'}, \tilde{H}(\hat{\mathbf{q}}_{T,g})) - r(\mathbf{p}_g, \hat{\mathbf{q}}_{T,g}) \right) \\ &= \sum_{g \in \mathcal{G}} \left(\max_{g' \in \mathcal{G}} \zeta(\mathbf{p}_{g'}, \hat{\mathbf{v}}_{T,g}) - r(\mathbf{p}_g, \hat{\mathbf{v}}_{T,g}) \right) \\ &\leq \sum_{g \in \mathcal{G}} \left(\left| \max_{g' \in \mathcal{G}} \zeta(\mathbf{p}_{g'}, \hat{\mathbf{v}}_{T,g}) - \max_{g' \in \mathcal{G}} \zeta(\mathbf{p}_{g'}, \tilde{\mathbf{v}}_{T,g}) \right| + |\tilde{r}_{T,g} - r(\mathbf{p}_g, \hat{\mathbf{v}}_{T,g})| \right) \\ &\leq \sum_{g \in \mathcal{G}} \left(L_\zeta \|\hat{\mathbf{v}}_{T,g} - \tilde{\mathbf{v}}_{T,g}\|_2 + |\tilde{r}_{T,g} - r(\mathbf{p}_g, \hat{\mathbf{v}}_{T,g})| \right) \\ &\leq (L_\zeta + \sqrt{N_{\mathcal{G}}}) \left\| \left[\begin{array}{c} \tilde{r}_{T,g} \\ \tilde{\mathbf{v}}_{T,g} \end{array} \right]_{g \in \mathcal{G}} - \frac{1}{T} \sum_{t=1}^T \underline{r}(I_t, J_t) \right\|_2 \\ &= (L_\zeta + \sqrt{N_{\mathcal{G}}}) \inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T \underline{r}(I_t, J_t) \right\|_2, \end{aligned}$$

where we denoted by $N_{\mathcal{G}}$ the cardinality of \mathcal{G} . Resorting to the convergence rate stated in Theorem 12 concludes the proof, up to the stated Lipschitzness of ζ , which we now prove.

It suffices to show that for all fixed elements $\mathbf{p} \in \Delta(\mathcal{I})$, the functions $\mathbf{v} \in \mathbb{R}_+^{\mathcal{J}} \mapsto \zeta(\mathbf{p}, \mathbf{v})$ are Lipschitz, with a Lipschitz constant that is independent of \mathbf{p} .

Consider two elements $\mathbf{v}, \mathbf{v}' \in \mathbb{R}_+^{\mathcal{J}}$. If $\|\mathbf{v}'\| = 0$ and $\|\mathbf{v}\|_1 > 0$, then

$$|\zeta(\mathbf{p}, \mathbf{v}) - \zeta(\mathbf{p}, \mathbf{v}')| = |\zeta(\mathbf{p}, \mathbf{v})| = \|\mathbf{v}\|_1 \left| \rho \left(\mathbf{p}, \tilde{H} \left(\frac{\mathbf{v}}{\|\mathbf{v}\|_1} \right) \right) \right| \leq R \|\mathbf{v}\|_1 = R \|\mathbf{v} - \mathbf{v}'\|_1 .$$

In the case where both \mathbf{v} and \mathbf{v}' are non zero,

$$\begin{aligned} & \zeta(\mathbf{p}, \mathbf{v}) - \zeta(\mathbf{p}, \mathbf{v}') \\ &= \|\mathbf{v}\|_1 \rho \left(\mathbf{p}, \tilde{H} \left(\frac{\mathbf{v}}{\|\mathbf{v}\|_1} \right) \right) - \|\mathbf{v}'\|_1 \rho \left(\mathbf{p}, \tilde{H} \left(\frac{\mathbf{v}'}{\|\mathbf{v}'\|_1} \right) \right) \\ &= \|\mathbf{v}\|_1 \left[\rho \left(\mathbf{p}, \tilde{H} \left(\frac{\mathbf{v}}{\|\mathbf{v}\|_1} \right) \right) - \rho \left(\mathbf{p}, \tilde{H} \left(\frac{\mathbf{v}'}{\|\mathbf{v}'\|_1} \right) \right) \right] + (\|\mathbf{v}\|_1 - \|\mathbf{v}'\|_1) \rho \left(\mathbf{p}, \tilde{H} \left(\frac{\mathbf{v}'}{\|\mathbf{v}'\|_1} \right) \right). \end{aligned}$$

Therefore, by using the Lipschitzness proved at the end of the proof of Corollary 13, by two applications of the triangle inequality, and by noting that $\|\cdot\|_2 \leq \|\cdot\|_1 \leq \sqrt{N_{\mathcal{J}}} \|\cdot\|_2$, we get

$$\begin{aligned} |\zeta(\mathbf{p}, \mathbf{v}) - \zeta(\mathbf{p}, \mathbf{v}')| &\leq \|\mathbf{v}\|_1 L_{\rho} \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|_1} - \frac{\mathbf{v}'}{\|\mathbf{v}'\|_1} \right\|_2 + \|\mathbf{v} - \mathbf{v}'\|_1 R \\ &\leq L_{\rho} \left\| \mathbf{v} - \mathbf{v}' + \left(1 - \frac{\|\mathbf{v}\|_1}{\|\mathbf{v}'\|_1} \right) \mathbf{v}' \right\|_2 + R \sqrt{N_{\mathcal{J}}} \|\mathbf{v} - \mathbf{v}'\|_2 \\ &\leq (L_{\rho} + R \sqrt{N_{\mathcal{J}}}) \|\mathbf{v} - \mathbf{v}'\|_2 + L_{\rho} \left| 1 - \frac{\|\mathbf{v}\|_1}{\|\mathbf{v}'\|_1} \right| \|\mathbf{v}'\|_2 \\ &\leq (L_{\rho} + R \sqrt{N_{\mathcal{J}}}) \|\mathbf{v} - \mathbf{v}'\|_2 + L_{\rho} \left| \|\mathbf{v}'\|_1 - \|\mathbf{v}\|_1 \right| \frac{\|\mathbf{v}'\|_2}{\|\mathbf{v}'\|_1} \\ &\leq (L_{\rho} + R \sqrt{N_{\mathcal{J}}}) \|\mathbf{v} - \mathbf{v}'\|_2 + L_{\rho} \|\mathbf{v} - \mathbf{v}'\|_1 \\ &\leq (L_{\rho} + (R + L_{\rho}) \sqrt{N_{\mathcal{J}}}) \|\mathbf{v} - \mathbf{v}'\|_2 . \end{aligned}$$

We therefore proved the required Lipschitzness, with constant $L_{\zeta} = L_{\rho} + (R + L_{\rho}) \sqrt{N_{\mathcal{J}}}$. ■

B Proofs of results extracted from other works

The proofs below reproduce arguments that were published elsewhere; we rewrite them with our notation for the convenience of the reviewers and to make our submission self-contained.

B.1 Proof of the basic approachability results

This material is provided only for the convenience of the reviewers; it is standard and can be found, e.g., in Cesa-Bianchi and Lugosi (2006, Section 7.7 and Exercise 7.23).

Proof:[of Theorem 3] Since c_t is the projection of \widehat{m}_t on the closed convex set \mathcal{C} with respect to the ℓ^2 -norm, the following geometric property is satisfied:

$$\forall c \in \mathcal{C}, \quad \langle \widehat{m}_t - c_t, c - c_t \rangle \leq 0.$$

By assumption, for every $\mathbf{y} \in \Delta(\mathcal{B})$, there exists $\mathbf{x} \in \Delta(\mathcal{A})$ such that $m(\mathbf{x}, \mathbf{y}) \in \mathcal{C}$; therefore, the above-stated geometric property implies that

$$\max_{\mathbf{y} \in \Delta(\mathcal{B})} \min_{\mathbf{x} \in \Delta(\mathcal{A})} \langle \widehat{m}_t - c_t, m(\mathbf{x}, \mathbf{y}) - c_t \rangle \leq 0.$$

By von Neumann's minimax theorem,

$$\max_{\mathbf{y} \in \Delta(\mathcal{B})} \min_{\mathbf{x} \in \Delta(\mathcal{A})} \langle \widehat{m}_t - c_t, m(\mathbf{x}, \mathbf{y}) - c_t \rangle = \min_{\mathbf{x} \in \Delta(\mathcal{A})} \max_{\mathbf{y} \in \Delta(\mathcal{B})} \langle \widehat{m}_t - c_t, m(\mathbf{x}, \mathbf{y}) - c_t \rangle \leq 0.$$

In view of the defining minimax choice of $\mathbf{x}_{t+1} \in \Delta(\mathcal{A})$, the above inequality yields that for all $\mathbf{z}_{t+1} \in \Delta(\mathcal{B})$,

$$\langle \widehat{m}_t - c_t, m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1}) - c_t \rangle \leq 0. \quad (14)$$

In the rest of the proof, we choose \mathbf{z}_{t+1} to be either \mathbf{y}_{t+1} or $\delta_{B_{t+1}}$, depending on whether mixed or pure actions are taken and observed; in particular, we have the rewriting

$$\widehat{m}_t = \frac{1}{t} \sum_{\tau=1}^t m(\mathbf{x}_\tau, \mathbf{z}_\tau).$$

Straightforward calculation show that

$$\begin{aligned} \widehat{m}_{t+1} &= \frac{1}{t+1} \sum_{\tau=1}^{t+1} m(\mathbf{x}_\tau, \mathbf{z}_\tau) \\ &= \frac{1}{t+1} m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1}) + \frac{1}{t} \sum_{\tau=1}^t m(\mathbf{x}_\tau, \mathbf{z}_\tau) - \frac{1}{t(t+1)} \sum_{\tau=1}^t m(\mathbf{x}_\tau, \mathbf{z}_\tau) \\ &= \widehat{m}_t + \frac{1}{t+1} (m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1}) - \widehat{m}_t). \end{aligned}$$

Denote by

$$d_t = \inf_{c \in \mathcal{C}} \|c - \widehat{m}_t\|_2 = \|c_t - \widehat{m}_t\|_2$$

the ℓ^2 -distance of \widehat{m}_t to \mathcal{C} . Now, for all $t \geq 1$,

$$\begin{aligned} d_{t+1}^2 &\leq \|c_t - \widehat{m}_{t+1}\|_2^2 = \left\| (c_t - \widehat{m}_t) + \frac{1}{t+1} (\widehat{m}_t - m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1})) \right\|_2^2 \\ &= \|c_t - \widehat{m}_t\|_2^2 + \frac{2}{t+1} \langle \widehat{m}_t - c_t, m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1}) - \widehat{m}_t \rangle \\ &\quad + \frac{\|\widehat{m}_t - m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1})\|_2^2}{(t+1)^2} \\ &= \left(1 - \frac{2}{t+1}\right) \underbrace{\|c_t - \widehat{m}_t\|_2^2}_{= d_t^2} + \frac{2}{t+1} \underbrace{\langle \widehat{m}_t - c_t, m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1}) - c_t \rangle}_{\leq 0 \text{ by (14)}} \\ &\quad + \frac{\|\widehat{m}_t - m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1})\|_2^2}{(t+1)^2}. \end{aligned}$$

By the triangle inequality,

$$\|\widehat{m}_t - m(\mathbf{x}_{t+1}, \mathbf{z}_{t+1})\|_2^2 \leq 4M^2;$$

thus, we proved that

$$d_{t+1}^2 \leq \left(1 - \frac{2}{t+1}\right) d_t^2 + \frac{4M^2}{t+1}.$$

Since

$$\left(1 - \frac{2}{t+1}\right) \frac{1}{t} + \frac{1}{(t+1)^2} = \frac{(t+1)^2 - 2(t+1) + t}{t(t+1)^2} = \frac{t^2 - 1 + t}{t(t+1)^2} \leq \frac{1}{t+1},$$

a simple induction argument yields that $d_T^2 \leq 4M^2/T$ for all $T \geq 1$; which concludes the proof in the case of mixed actions taken and observed.

In the case of pure actions taken and observed, we need an additional concentration argument. We denote by \mathbb{E}_t the conditional expectation at round t with respect to B_t and the \mathbf{x}_s, A_s, B_s , where $1 \leq s \leq t-1$; we have

$$\mathbb{E}_t[m(A_t, B_t)] = m(\mathbf{x}_t, B_t) = m(\mathbf{x}_t, \mathbf{z}_t).$$

In addition, the quantities $m(\mathbf{x}_t, \mathbf{z}_t) - m(A_t, B_t)$ are bounded in norm by $2M$. By the version of the Hoeffding-Azuma inequality for sums of Hilbert space-valued martingale differences already used in the proof of Theorem 12, we therefore have that for all $T \geq 1$, with probability at least $1 - \delta$,

$$\left\| \frac{1}{T} \sum_{t=1}^T m(A_t, B_t) - \frac{1}{T} \sum_{t=1}^T m(\mathbf{x}_t, \mathbf{z}_t) \right\|_2 \leq 4M \sqrt{\frac{\ln(2/\delta)}{T}},$$

which, combined with the deterministic bound on d_T , entails, still with probability at least $1 - \delta$,

$$\inf_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^T m(A_t, B_t) \right\|_2 \leq \frac{2M}{\sqrt{T}} \left(1 + 2\sqrt{\ln(2/\delta)}\right).$$

This concludes the proof. ■

B.2 Proof of a concentration argument

We re-prove here the inequality (13), that is directly extracted from Lugosi et al. (2008, Section 6). Again, this is only for the sake of self-containment.

Proof: For all $(i, j) \in \mathcal{I} \times \mathcal{J}$, the quantity $H(i, j)$ is a probability distribution over \mathcal{H} ; we denote by $H_s(i, j)$ the probability mass that it puts on some element $s \in \mathcal{H}$.

We consider a fixed block n . Equation (6) indicates that for each pair $(i, s) \in \mathcal{I} \times \mathcal{H}$,

$$\sum_{t=(n-1)L+1}^{nL} \left(\frac{\mathbb{I}_{\{S_t=s\}} \mathbb{I}_{\{I_t=i\}}}{p_{I_t,n}} - H_s(i, J_t) \right)$$

is a sum of L elements of a martingale difference sequence. The conditional variances of the increments are bounded by

$$\mathbb{E}_t \left[\left(\frac{\mathbb{I}_{\{S_t=s\}} \mathbb{I}_{\{I_t=i\}}}{p_{I_t,n}} \right)^2 \right] \leq \frac{1}{p_{i,n}^2} \mathbb{E}_t [\mathbb{I}_{\{I_t=i\}}] = \frac{1}{p_{i,n}};$$

since by definition of the strategy, $\mathbf{p}_n = (1 - \gamma) \mathbf{x}_n + \gamma \mathbf{u}$, we have that $p_{i,n} \geq \gamma/N_{\mathcal{I}}$, which shows that the sum of the conditional variances is bounded by

$$\sum_{t=(n-1)L+1}^{nL} \text{Var}_t \left(\frac{\mathbb{I}_{\{S_t=s\}} \mathbb{I}_{\{I_t=i\}}}{p_{I_t,n}} \right) \leq \frac{LN_{\mathcal{I}}}{\gamma}.$$

The Bernstein-Freedman inequality (see Freedman 1975 or Cesa-Bianchi et al. 2006, Lemma A.1) therefore indicates that with probability at least $1 - \delta$,

$$\left| \frac{1}{L} \sum_{t=(n-1)L+1}^{nL} \frac{\mathbb{I}_{\{S_t=s\}} \mathbb{I}_{\{I_t=i\}}}{p_{I_t,n}} - \underbrace{\frac{1}{L} \sum_{t=(n-1)L+1}^{nL} H_s(i, J_t)}_{= H_s(i, \widehat{\mathbf{q}}_n) \text{ by (7)}} \right| \leq \sqrt{2 \frac{N_{\mathcal{I}}}{\gamma L} \ln \frac{2}{\delta}} + \frac{1}{3} \frac{N_{\mathcal{I}}}{\gamma L} \ln \frac{2}{\delta}.$$

Therefore, by summing the above inequalities over $i \in \mathcal{I}$ and $s \in \mathcal{H}$, we get (after a union bound) that with probability at least $1 - N_{\mathcal{I}}N_{\mathcal{H}}\delta$,

$$\left\| \tilde{\sigma}_n - \tilde{H}(\hat{\mathbf{q}}_n) \right\|_2 \leq \sqrt{N_{\mathcal{I}}N_{\mathcal{H}}} \left(\sqrt{\frac{2N_{\mathcal{I}}}{\gamma L} \ln \frac{2}{\delta}} + \frac{1}{3} \frac{N_{\mathcal{I}}}{\gamma L} \ln \frac{2}{\delta} \right).$$

Finally, since $\hat{\sigma}_n$ is the projection in the ℓ^2 -norm of $\tilde{\sigma}_n$ onto the convex set \mathcal{F} , to which $\tilde{H}(\hat{\mathbf{q}}_n)$ belongs, we have that

$$\left\| \hat{\sigma}_n - \tilde{H}(\hat{\mathbf{q}}_n) \right\|_2 \leq \left\| \tilde{\sigma}_n - \tilde{H}(\hat{\mathbf{q}}_n) \right\|_2,$$

and this concludes the proof. ■