

Note on Max Lin-2 above Average

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Abstract

In the Max Lin-2 problem we are given a system S of m linear equations in n variables over \mathbb{F}_2 in which Equation j is assigned a positive integral weight w_j for each j . We wish to find an assignment of values to the variables which maximizes the total weight of satisfied equations. This problem generalizes Max Cut. The expected weight of satisfied equations is $W/2$, where $W = w_1 + \dots + w_m$; $W/2$ is a tight lower bound on the optimal solution of Max Lin-2.

Mahajan et al. (J. Comput. Syst. Sci. 75, 2009) stated the following parameterized version of Max Lin-2: decide whether there is an assignment of values to the variables that satisfies equations of total weight at least $W/2 + k$, where k is the parameter. They asked whether this parameterized problem is fixed-parameter tractable, i.e., can be solved in time $f(k)(nm)^{O(1)}$, where $f(k)$ is an arbitrary computable function in k only. Their question remains open, but using some probabilistic inequalities and, in one case, a Fourier analysis inequality, Gutin et al. (IWPEC 2009) proved that the problem is fixed-parameter tractable in three special cases.

In this paper we significantly extend two of the three special cases using only tools from combinatorics. We show that one of our results can be used to obtain a combinatorial proof that another problem from Mahajan et al. (J. Comput. Syst. Sci. 75, 2009), Max r -SAT above the Average, is fixed-parameter tractable for each $r \geq 2$. Note that Max r -SAT above the Average has been already shown to be fixed-parameter tractable by Alon et al. (SODA 2010), but the paper used the approach of Gutin et al. (IWPEC 2009).

Keywords: algorithms; fixed-parameter tractability; Max Lin; Max Sat.

1 Introduction

A *parameterized problem* is a subset $L \subseteq \Sigma^* \times \mathbb{N}$ over a finite alphabet Σ . L is *fixed-parameter tractable* if the membership of an instance (x, k) in $\Sigma^* \times \mathbb{N}$ can be decided in time $f(k)|x|^{O(1)}$ where f is a computable function of the *parameter* k only [2, 3, 11]. If the nonparameterized version of L (where k is just a part of the input) is NP-hard, then the function $f(k)$ must be superpolynomial provided $P \neq NP$. Often $f(k)$ is moderately exponential, which makes the problem practically tractable for small values of k . Thus, it is important to parameterize a problem in such a way that the instances with small values of k are of interest.

Consider the following well-known problem: given a connected digraph $D = (V, A)$, find an acyclic subdigraph of D with the maximum number of arcs. We can parameterize this problem in the standard way by asking whether D contains an acyclic subdigraph with at least k arcs. It is easy to prove that this parameterized problem is fixed-parameter tractable by observing that D always has an acyclic subdigraph with at least $|A|/2$ arcs. Indeed, if $k \leq |A|/2$, the answer is YES and if $k > |A|/2$ then $|V| \leq |A| + 1 \leq 2k$. In the last case, we can check whether D has an acyclic subdigraph with at least k arcs by generating all $|V|! \leq (2k)!$ orderings of V and constructing subdigraphs of D induced by forward arcs. This gives an $|A|^{O(1)}(2k)!$ -time algorithm. However, this algorithm is impractical as $k > |A|/2$ is large when $|A|$ is large.

Note that $|A|/2$ is a tight lower bound on the solution; indeed, $|A|/2$ is the optimum for all digraphs in which the existence of an arc implies the existence of the opposite arc. Thus, the following parametrization above a tight lower bound is more appropriate: decide whether D contains an acyclic subdigraph with at least $|A|/2 + k$ arcs.

Mahajan and Raman [9] were the first to consider problems parameterized above tight lower bounds (PATLB). They indicated that such parameterizations are often the only ones of practical value. Mahajan et al. [10] proved several results for problems PATLB, and noted that the parameterized complexity of only a few such problems was investigated in the literature (partially, because this is often a challenging question) and stated several open questions on the topic. Apart from [9, 10], until very recently there were only three papers on problems PATLB: Gutin et al. [5], Gutin et al. [6] and Heggernes et al. [8].

The paper of Mahajan et al. [10] triggered several recent papers where some of the questions in [10] were solved. In particular, Gutin et al. [4] proved that the above-mentioned maximum acyclic subdigraph problem PATLB is fixed-parameter tractable and so are three special cases of Max Lin-2 PATLB (the last problem is defined in the next section). In their proofs, Gutin et al. [4] used some probabilistic inequalities and, in one case, a Fourier analysis inequality. Using this approach, Alon et al. [1] proved that Max r -SAT PATLB as well as many other Boolean Constraint Satisfaction Problems PATLB are fixed-parameter tractable. The parameterized complexity of Max r -SAT PATLB was one of the central open problems in Mahajan et al. [10].

In this short paper, we significantly extend two of the three special cases of Max Lin-2 PATLB using only tools from combinatorics (see Theorems 1 and 2). We show that our extensions cannot be obtained by the approach of [4]. We also show that the Boolean Constraint Satisfaction Problems PATLB results of [1] can be proved by combinatorial arguments only, using one of our results for Max Lin-2 PATLB.

2 Max $r(n)$ -Lin-2 above the Average

Consider the following problem for a fixed function $r(n)$. This problem should be called Max $r(n)$ -Lin-2 PATLB, if we follow [4], but the new name appears to be clearer and simpler.

MAX $r(n)$ -LIN-2 ABOVE THE AVERAGE (or MAX $r(n)$ -LIN-2 AA for short)

Instance: A system S of m linear equations in n variables over \mathbb{F}_2 , where no equation has more than $r = r(n)$ variables and Equation j is assigned a positive integral weight w_j , $j = 1, \dots, m$, and a nonnegative integer k . We will write Equation j in S as $\sum_{i \in \alpha_j} z_i = b_j$, where $\alpha_j \subseteq \{1, 2, \dots, n\}$ and $|\alpha_j| \leq r$.

Parameter: The integer k .

Question: Is there an assignment of values to the n variables such that the total weight of the satisfied equations is at least $(W + k)/2$, where $W = w_1 + \dots + w_m$?

We assume that each of the n variables appears in at least one equation of S and no equation has an empty left-hand side.

Note that $W/2$ is indeed a tight lower bound for the above problem, as the expected weight of satisfied equations in a random assignment is $W/2$, and no assignment of values to the variables satisfies equations of total weight more than $W/2$ if S consists of pairs of equations with identical left-hand sides and contradicting right-hand sides.

Consider two reduction rules for MAX $r(n)$ -LIN-2 AA introduced in [4].

Reduction Rule 1. *Let A be the matrix of the coefficients of the variables in S , let $t = \text{rank}A$ and let columns a^{i_1}, \dots, a^{i_t} of A be linearly independent. Then delete all variables not in $\{z_{i_1}, \dots, z_{i_t}\}$ from the equations of S .*

Reduction Rule 2. *If we have, for a subset α of $\{1, 2, \dots, n\}$, an equation $\sum_{i \in \alpha} z_i = b'$ with weight w' , and an equation $\sum_{i \in \alpha} z_i = b''$ with weight w'' , then we replace this pair by one of these equations with weight $w' + w''$ if $b' = b''$ and, otherwise, by the equation whose weight is bigger, modifying its new weight to be the difference of the two old ones. If the resulting weight is 0, we delete the equation from the system.*

Lemma 1. [4] *Let T be obtained from S by Rule 1 or 2. Then T is a YES-instance if and only if S is a YES-instance. Moreover, T can be obtained from S in time polynomial in n and m .*

We cannot change S using Rule 1 (Rule 2), S is *irreducible* by Rule 1 (Rule 2). If S is irreducible by Rule 1, we have $n \leq m$. If S is irreducible by Rule 2, the symmetric difference $\alpha_j \Delta \alpha_p \neq \emptyset$ for each pair $j \neq p$.

Consider the following algorithm for MAX $r(n)$ -LIN-2 AA, which is a modification of an algorithm used in [7]. We assume that, in the beginning, no equation or variable in S is marked.

ALGORITHM \mathcal{A}

While $S \neq \emptyset$ and less than k equations are marked, do the following:

1. For $1 \leq i \leq n$, calculate ρ_i , the number of equations in S containing z_i .
2. Choose z_l with minimum ρ_l among all variables still in S . Mark z_l .
3. Choose an arbitrary equation containing z_l , $\sum_{i \in \alpha} z_i = b$.
4. Mark this equation and delete it from S .
5. Replace every equation $\sum_{i \in \alpha'} z_i = b'$ in S containing z_l by $\sum_{i \in \alpha \Delta \alpha'} z_i = b''$, where $b'' = b + b'$.
6. Apply Rule 2. (As a result, several equations can be of weight 0 and, thus, are deleted from the system.)

Observe that \mathcal{A} runs in polynomial time. We have the following simple yet important property of \mathcal{A} .

Lemma 2. *If the input system S is irreducible by Rule 2 and algorithm \mathcal{A} has marked k equations in S , then S is a YES-instance.*

Proof. Assume that \mathcal{A} has marked k equations in the input system S and let T be the system of equations remained in S after \mathcal{A} has stopped. Observe that for every assignment of values to the variables z_1, \dots, z_n that satisfies all marked equations, the operation of Step 5 of \mathcal{A} replaces S by an *equivalent* system (i.e., both systems have the same difference in weight of satisfied and falsified equations). Thus, for every such assignment, S is equivalent to T together with the marked equations. We will show that there is an assignment that satisfies all marked equations and half of equations of T (in terms of weight). This will be sufficient due to the following. Let W' be the total weight of the marked equations. Then the total weight of the satisfied equations is $W' + (W - W')/2 = (W + W')/2 \geq (W + k)/2$ since $W' \geq k$ by integrality of the weights.

We can find a required assignment as follows. We start by finding an assignment of values to the variables in T that satisfies half of equations of T (in terms of weight), using the following algorithm from [7]: Assign values to the variables sequentially, and after each assignment, perform the obvious algebraic simplifications. When about to assign a value to x_j , consider all equations of the form $x_j = b$, for constant b . Assign x_j a value satisfying at least half of these equations (in terms of weight).

It remains to assign any values to the variables not in T of the marked equations such that they are all satisfied. This is possible if we find an assignment that satisfies the last marked equation, then find an assignment satisfying the equation marked before the last, etc. Indeed, the equation marked before the last contains a (marked) variable z_l not appearing in the last equation, etc. \square

Lemma 3. *If an instance of MAX $r(n)$ -LIN-2 AA is irreducible by Rule 2 and its number of variables $n \geq 2^k r(n)$, then it is a YES-instance.*

Proof. Let ρ_{l_t} be the ρ_l picked in step 2 of Iteration t of algorithm \mathcal{A} , and let R_t be the maximum number of variables in any equation in S at Iteration t . Observe that $R_1 = r$, and that $R_{t+1} \leq 2R_t$. Thus, $R_t \leq 2^{t-1}r$. In Iteration t of \mathcal{A} , by minimality of ρ_{l_t} , at most $(2\rho_{l_t} - 1)R_t/\rho_{l_t} < 2R_t$ variables will be removed from the system. Thus, the total number of variables completely deleted from the system after $k - 1$ iterations is less than $\sum_{t=1}^{k-1} 2R_t \leq \sum_{t=1}^{k-1} 2^t r < 2^k r$. So, if $2^k r \leq n$ then Iteration k is possible, and hence, by Lemma 2, we have a YES-instance. \square

Theorem 1. *If an instance of MAX $r(n)$ -LIN-2 AA is irreducible by Rule 2 and $r(n) = o(n)$, then MAX $r(n)$ -LIN-2 AA is fixed-parameter tractable.*

Proof. Let $r = o(n)$. By Lemma 3, if $n \geq 2^k r$, then we have a YES-instance. Otherwise, $n < 2^k r$ and so $n \leq g(k)$ for some function $g(k)$ depending on k only. In the last case, in time $O(m^{O(1)} 2^{g(k)})$ we can check whether our instance is a YES-instance. \square

Gutin et al. [4] prove that MAX $r(n)$ -LIN-2 AA is fixed-parameter tractable for $r = O(1)$. Using the method of [4] one can only extend this result to $r = o(\log m)$. If $r = o(\log m)$ then $r = o(n)$ (since $m < 2^n$ by Rule 2) and, thus, MAX $r(n)$ -LIN-2 AA is fixed-parameter tractable by Theorem 1. However, if $r = \Omega(\log m)$ and $r = o(n)$ then MAX $r(n)$ -LIN-2 AA is fixed-parameter tractable by Theorem 1, but this result cannot be obtained using the method of [4].

Let $\rho = \max_{1 \leq i \leq n} \rho_i$. Let MAX LIN-2 AA be MAX $r(n)$ -LIN-2 AA with $r(n) = n$.

Theorem 2. *Let the input system S be irreducible by Rules 1 and 2 and let S have m equations. If $\rho = o(m)$, then MAX LIN-2 AA is fixed-parameter tractable.*

Proof. Let $\rho = o(m)$. Apply algorithm \mathcal{A} (for this theorem, there is no need to do Step 1 or select the z_l with minimum ρ_l on Step 2; we can arbitrarily choose any z_l still in S). We will show that after $k - 1$ iterations at most $2\rho(k - 1)$ equations have been deleted. Let Q be the set of equations that, at the beginning, contain at least one of $z_{l_1}, \dots, z_{l_{k-1}}$, where $z_{l_1}, \dots, z_{l_{k-1}}$ are the variables marked in the first $k - 1$ iterations. Note that $|Q| \leq \rho(k - 1)$. An equation not in Q is only deleted if there exists an equation in Q such that, after some applications of the symmetric difference operation of Step 5, the two equations have the same left-hand side. Furthermore, observe that each equation in Q can only ever have the same left-hand side as at most one equation not in Q . So the number of equations removed is at most $2|Q| \leq 2\rho(k - 1)$. Observe that either $2\rho(k - 1) < m$ in which case Iteration k is possible and we can apply Lemma 2, or $m \leq 2\rho(k - 1)$ and $m \leq f(k)$ for some function $f(k)$ depending on k only. If $m \leq f(k)$, $n \leq m \leq f(k)$ and in time $O(m^{O(1)} 2^{f(k)})$ we can check whether our instance is a YES-instance. \square

Using the approach of [4] it is easy to show that if $\rho = o(\sqrt{m})$ then MAX LIN-2 AA is fixed-parameter tractable and this cannot be extended even to the case $\rho = \Theta(\sqrt{m})$. Thus, Theorem 2 provides a much stronger result.

3 Boolean Constraint Satisfaction Problems above Average

The aim of this section is to show that in the proofs of the main results of [1] for a wide family of Boolean Constraint Satisfaction Problems above the Average, Lemma 3 can replace probabilistic and Fourier analysis inequalities. As a result, the proofs become purely combinatorial and slightly simpler. Alon et al. [1] provide all details for MAX r -SAT AA (defined below) only and comment that basically the same arguments can be used for a wide class of Boolean Constraint Satisfaction Problems above the Average. Thus, we restrict ourselves to MAX r -SAT AA only as basically the same arguments can be used for the wide class of Boolean Constraint Satisfaction Problems above the Average.

Let $r(\geq 2)$ be a constant.

MAX r -SAT ABOVE THE AVERAGE (or MAX r -SAT AA for short)

Instance: A pair (F, k) where F is a multiset of m clauses, each of size r ; F contains only variables x_1, x_2, \dots, x_n , and k is a nonnegative integer.

Parameter: The integer k .

Question: Is there a truth assignment to the n variables such that the total number of satisfied clauses is at least $E + k2^{-r}$, where $E = m(1 - 2^{-r})$, the average number of satisfied clauses?

Let F contain clauses C_1, \dots, C_m in the variables x_1, x_2, \dots, x_n . We may assume that $x_i \in \{-1, 1\}$, where -1 corresponds to TRUE. For F , consider a polynomial

$$X = \sum_{j=1}^m (1 - \prod_{x_i \in C_j} (1 + \epsilon_i x_i)),$$

where $\epsilon_i \in \{-1, 1\}$ and $\epsilon_i = 1$ if and only if x_i is in C .

Lemma 4. [1] *The answer to MAX r -SAT AA is YES if and only if there exists an assignment for x_1, x_2, \dots, x_n for which $X \geq k$.*

Theorem 3. *The problem MAX r -SAT AA is fixed-parameter tractable for each constant $r \geq 2$.*

Proof. Given a MAX r -SAT AA instance, define the polynomial X of degree at most r as above. After algebraic simplification $X = X(x_1, x_2, \dots, x_n)$ can be written as $X = \sum_{I \in \mathcal{S}} X_I$, where $X_I = c_I \prod_{i \in I} x_i$, each c_I is a nonzero integer and \mathcal{S} is a family of nonempty subsets of $\{1, \dots, n\}$ each with at most r elements. Thus, X is a polynomial of degree at most r .

Now define an instance MAX r -LIN-2 AA with the variables z_1, z_2, \dots, z_n as follows. For each nonzero term $c_I \prod_{i \in I} x_i$ consider the linear equation $\sum_{i \in I} z_i = b$, where $b = 0$ if c_I is positive, and $b = 1$ if c_I is negative, and assign this equation the weight $w_I = |c_I|$. It is easy to check that this system of equations has an assignment z_i satisfying equations of total weight at least $[\sum_{I \in \mathcal{S}} w_I + k]/2$ if and only if there are $x_i \in \{-1, 1\}$ so that $X(x_1, x_2, \dots, x_n) \geq k$. This is shown by the transformation

$x_i = (-1)^{z_i}$. Let n' be the number of variables in the instance of MAX r -LIN-2 AA; clearly $n' \leq n$. Observe that $|S| \leq n^r$.

By Lemma 3, if $n' > 2^k r$, then we have a YES-instance of MAX r -LIN-2 AA and, thus, by Lemma 4, the answer to MAX r -SAT AA is YES. If $n' \leq 2^k r$ then we can find the maximum of X by using all assignments in time $|S|^{O(1)} 2^{n'} = n^{O(r)} 2^{r 2^k}$ and apply Lemma 4 to check whether the answer to MAX r -SAT AA is YES.

It remains to observe that the instance of MAX r -LIN-2 AA can be constructed in time $(m 2^r)^{O(1)}$. \square

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