Dutch Books and Combinatorial Games

Peter Harremoës
Centrum Wiskunde & Informatica (CWI), The Netherlands
P.Harremoes@cwi.nl

Abstract

The theory of combinatorial game (like board games) and the theory of social games (where one looks for Nash equilibria) are normally considered as two separate theories. Here we shall see what comes out of combining the ideas. The central idea is Conway's observation that real numbers can be interpreted as special types of combinatorial games. Therefore the payoff function of a social game is a combinatorial game. Probability theory should be considered as a safety net that prevents inconsistent decisions via the Dutch Book Argument. This result can be extended to situations where the payoff function is a more general game than a real number. The main difference between number valued payoff and game valued payoff is that a probability distribution that gives nonnegative mean payoff does not ensure that the game will be lost due to the existence of infinitisimal games. Also the Ramsay/de Finetti theorem on exchangable sequences is discussed.

Keywords. Combinatorial game, Dutch book theorem, exchangable sequences, game theory, surreal number.

1 Introduction

The word game in mathematics has two different meanings. The first type of games are the social games where a number of agents ' at the same time have to make a choice and where the payoff to each agent is a function of all agents choices. Each agent has his own payoff function. The question is how the agents should choose in order to maximize their own payoff. In general the players may benefit by making coalitions against each other. This kind of game theory has found important applications in social sciences and economy. A special class of these social games are the two-persons zero-sum games where collaboration between the agents makes no sense.

The second type of games are combinatorial games. These are mathematical models of board games. These games are the ones that people find interesting and amusing. Games that people do for amusement often involve an element of chance, generated by, for instance, dice, but the combinatorial games are by definition the ones that do not contain this element. Therefore they are sometimes called *games of no chance* [24]. Examples from this category are chess, nim, nine-mens-morris, and go. Combinatorial game theory has been particularly successful in the analysis of impartial games like nim [4,5] and has lead to a better understanding of endgames in go [2,3,24].

Social games and combinatorial games are build on very different ideas and many scientists only know one of the types of game theory. Therefore we will give a short introduction to each of these types of game theory and also fix notation. After these introductory sections we shall combine the ideas from the two branches of game theory and see how such a combined view effects our understanding of probabilities.

As we shall see our probability distributions will not always be real valued. In our approach the focus is on the order structure and its realtion to decision theory. A somewhat orthogonal approach was taken in [17] where the probabilities were a metric space that was not ordered.

2 Social games

Here some basic concepts and facts about social about games are presented. The readers who are interested in a deeper understanding of the theory of social games should consult [32] for an easy introduction or [13] for a more detailed exposition. The book [1] describes convex optimization with many technical details. The theory of two persons zero sum games was founded by J. von Neumann together with O. Morgenstern [33].

g	s_1	s_2
b_1	1	0
b_2	0	-1

Table 1: Payoff for Alice.

g	Rock	Scissors	Paper
Rock	0	-1	1
Scissors	1	0	-1
Paper	-1	1	0

Table 2: Payoff for Alice.

A social game with n players is described by n sets of *strategies*

$$S_1, S_2, ..., S_n$$

such that Player i can choose strategies from S_i. If Player i chooses strategy $s_i \in S_i, i = 1, 2, ..., n$ then Player j will get the *payoff* $g_j(s_1, s_2, ..., s_n)$ where g_j is a function from $S_1 \times S_2 \times ...S_n$ to $\mathbb{R}$.

A two persons zero sum game is a game with two players where $g_1 = -g_2$. In the case of two person zero-sum games we call the players *Alice* and *Bob* and denote their sets of strategies by A and B. Alice and Bob will never collaborate in a zero-sum game because what is good for one of the players is equally bad for the other.

A set of strategies $(s_1, s_2, ..., s_n)$ is Pareto optimal if no other set of strategies $(b_1, b_2, ..., b_n)$ exist such for any player the payoff is at least as good as the payoff given by $(s_1, s_2, ..., s_n)$ and at least one player gets a better payoff. A set of strategies $(s_1, s_2, ..., s_n)$ is called a *Nash equilibrium* if no player will benefit by changing his own strategy if the other players leave their strategies unchanged.

If a game has a unique Nash pair and both players are *rational*, then both players should play according to the Nash equilibrium. If one of the players is not rational the other may act better than according to the Nash equilibrium, but to take advantage of an other players lack of being rational may require quite a lot of psychological insight.

Example 1 *Consider a game with g given by Table 1.*

The pair (s_1, b_2) is a Nash pair.

Example 2 *Consider the popular game Rock, Scissor and Paper with g given by Table 2.*

If Alice chooses Rock Bob will chose Paper. But if Bob chooses Paper it would be better for Alice to choose Scissors. If Alice then chooses Scissors Bob would choose Rock and then Alice should play Paper. Then Bob should choose Scissors and Alice should choose Rock ... etc. There is no Nash equilibrium but any pair of strategies is Pareto optimal.

Assume that the players are allowed to use mixed strategies, i.e. chose independent probability distributions over the strategies. Let P_i be the mixed strategy of Player i. Then the mean payoff is

$$\sum_{(s_1,...,s_n)} g_j(s_1, s_2, ..., s_n) P_1(s_1) P_2(s_2) \cdot ... \cdot P_n(s_n).$$

This number is considered as the payoff of the game where mixed strategies are allowed.

Theorem 3 *Let S_i be finite sets and g_i be a payoff functions. If all players are allowed to use mixed strategies, then the game has a Nash equilibrium.*

There exists various different proofs of the existence of the existence of a Nash equilibria for two-person zero-sum games [1, 13, 32, 33]. In this note we shall focus on its equivalence with the Dutch Book Theorem and prove an extended version involving functions with surreal values in Section 7.

If Bob knows that Alice chooses strategy a then Bob will choose a strategy which achieves

$$\min_{b \in B} g(a, b).$$

Similarly, if Alice knows that Bob chooses strategy b then Alice chooses a strategy which achieves

$$\max_{a \in A} g(a, b).$$

These quantities satisfy

$$\min_{b \in B} g(a, b) \leq g(a, b).$$

Taking maximum over $a \in A$ gives

$$\max_{a \in A} \min_{b \in B} g(a, b) \leq \max_{a \in A} g(a, b).$$

Taking minimum over $b \in B$ gives

$$\max_{a \in A} \min_{b \in B} g(a, b) \leq \min_{b \in B} \max_{a \in A} g(a, b).$$

The game is said to be in equilibrium if these quantities are equal. The common value is called the *value of the game.*

Example 4 *In the social game Stone, Scissors and Paper*

$$\min_{b \in B} g(a_i, b) = -1$$

and

$$\max_{a \in A} g(a, b_j) = 1.$$

horse	h_1	h_2	h_3
odds	6	6	2

Table 3: Superfair odds

horse	h_1	h_2	h_3
odds	3	3	2

Table 4: Subfair odds

Thus,

$$\min_{b\in B}\max_{a\in A} g\left(a,b_j\right)=1$$

and

$$\max_{a\in A}\min_{b\in B} g\left(a,b\right)=-1$$

and the game is not in equilibrium. If mixed strategies are allowed there is a unique Nash equilibrium with the uniform distributions as optimal strategies for both players. Then the value of the game equals zero.

If a two-persons zero-sum game has a Nash equilibrium pair $\left(a^*,b^*\right)$, which is always the case if A and B are finite, then

$$\sup_{a\in A} g\left(a,b^*\right)=g\left(a^*,b^*\right)$$

and therefore

$$\inf_{b\in B}\sup_{a\in A} g\left(a,b\right)\leq g\left(a^*,b^*\right).$$

Similarly

$$\sup_{a\in A}\inf_{b\in B} g\left(a,b\right)\geq g\left(a^*,b^*\right).$$

Thus, the game is in equilibrium and the value of the game is $g\left(a^*,b^*\right)$. In particular all Nash equilibria have the same value.

3 Dutch books and Zero-sum games

We shall see how probabilities emerge in a natural way when we want to avoid making incoherent decisions in a complex environment.

Example 5 *A bookmaker has odds for the different horses in a horse race. Consider the odds in Table 3.*

The odds 6 here means 6-for-1. That is, the gambler puts down 1 £ on horse h_1 before the game and gets 6 £ if the horse wins. Otherwise the stake is lost. This should not be confused with odds 6-to-1 which means that the gambler will receive 7 £ and thus has won 6 £ when the stake of 1 £ is subtracted. Consider a gambler who knows the horses and think that these odds are very favorable. He has 5 £ and has to decide how to use the money. If all the money is put on one horse there is a good chance of getting a good payoff, but there is still the risk of losing all the money. Instead of putting all the money on a single horse the gambler puts 1 £ on each of the horses h_1 and h_2 and 3 £ on horse h_3. Then the gambler will always win 6 £. Thus the bookmaker will loose 1£ independently of the winner. This is called a Dutch book and the odds are said to be super-fair.

To avoid Dutch books the bookmaker should change the odds. The bookmaker assigns the probability vector $(1/4, 1/4, 1/2)$ to the horses and offers odds according to Table 4.

If somebody puts x £ on horse h_1 then the gambler's mean payoff is $\frac{1}{4}\cdot 3x+\frac{1}{4}\cdot 0+\frac{1}{2}\cdot 0=\frac{3x}{4}$. Thus the mean payoff of the bookmaker is $x/4>0$. Similarly the mean payoff is positive if somebody puts money on horse h_2 or h_3. Thus the mean payoff of the bookmaker is also positive if the gamblers have distributed their money in some way over the horses. Therefore a Dutch book for the bookmaker is not possible because it will have negative mean for any probability vector. If a gambler puts 1 £ on each horse the gambler has made a Dutch book because the gambler will never get a positive payoff.

Here we will introduce a version of the *Dutch Book Theorem*[1]. The Dutch Book Theorem was first formulated and proved by F. P. Ramsay in 1926 (reprinted in [29]) and later independently by B. de Finetti [9]. Alice wish to make a bet of the outcome. A bookmaker $b\in B$ offers the payoff $g\left(a,b\right)$ (positive or negative) if the outcome of a random event is $a\in A$. Thus $\left(a,b\right)\to g\left(a,b\right)$ can be considered as a matrix when A and B are finite sets. Alice should reject to play with some of the bookmakers b if Alice thinks that the payoff function $a\to g\left(a,b\right)$ is not favorable. For simplicity we shall assume that Alice accepts the payoff functions offered by the bookmakers $b\in B$.

By a *portfolio* we shall mean a probability vector $Q=\left(q_b\right)_{b\in B}$ on B according to which the money (and the risk) is distributed. Such a portifolio is described by the payoff function

$$\sum_{b\in B} q_b\cdot g\left(\cdot,b\right), \tag{1}$$

A *Dutch book* is a portifolio such that (1) is negative for all $a\in A$, i.e. the portifolio will gives Alice a

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[1] The term Dutch book is English and reflects that The Netherlands got much of its wealth by sailing and trading. Successful traders are often considered as greedy, but the author who now lives in The Netherlands and comes from Denmark, which has also based an important part of the country's wealth on sailing and trading, obviously does not share this view.

loss whatever happens. Assume that the amount of money K is invested according to a portifolio Q and that there extsts a Dutch book Q'. If Q has B as support then $q_{\min} = \min_{b\in B} q_b > 0$ and the payoff is

$$K\sum_{b\in B} q_b \cdot g(\cdot, b) =$$

$$K\sum_{b\in B}(q_b - q_{\min}\cdot q'_b)\cdot g(\cdot,b) + Kq_{\min}\sum_{b\in B} q'_b \cdot g(\cdot,b) <$$

$$K\left(1 - q_{\min}\cdot\sum_{b\in B} q'_b\right)\sum_{b\in B}\frac{(q_b - q_{\min}\cdot q'_b)\cdot g(\cdot,b)}{\sum_{b\in B}(q_b - q_{\min}\cdot q'_b)}.$$

Therefore it is better to invest $K\left(1 - q_{\min}\cdot\sum_{b\in B} q'_b\right)$ according to the porfolio $\frac{(q_b - q_{\min}\cdot q'_b)}{\sum_{b\in B}(q_b - q_{\min}\cdot q'_b)}$ instead of investing all the money according to Q. Here we have a tacit assumption that it is possible not to invest all money in the game and that these uninvested money will safe. This Dutch Book Argument means that Alice should reject to play with some of the bookmakers. If no Dutch book exists the set of payoff functions is said to be *coherent.*

Theorem 6 (Dutch Book Theorem) *If the set of payoff functions $g(\cdot,b), b \in B$ is coherent then there exists a probability vector $P = (p_a)_{a\in A}$ on A, such that*

$$\sum_{a\in A} p_a \cdot g(a,\cdot) \geq 0$$

for all $b \in B$.

Proof of equivalence of Th. 6 and Thm. 3. Assume that any two person zero-sum game has an equilibrium with value λ and optimal strategies P and Q. Then it is straight forward to check that $\lambda < 0$ leads to the first case in the Dutch book theorem and $\lambda \geq 0$ leads to the second case.

Assume that the Dutch book theorem holds. Consider a one-sided bet where the opponent is offered $f(a,b) = g(a,b) - \lambda$ if the opponent places his money at bookmaker a and b happens. According to the Dutch Book Theorem there exists a probability distribution P on A

$$\sum_{a\in A} p_a \cdot f(a,b) \leq 0$$

for all $b \in B$; or there exists a probability distribution Q on B such that

$$\sum_{b\in B} q_b \cdot f(a,b) \geq 0$$

for all $a \in A$. Therefore for all $\lambda \in \mathbb{R}$ there exists a probability distribution P on A such that

$$\sum_{a\in A} p_a \cdot g(a,b) \leq \lambda \tag{2}$$

for all $a \in A$ or there exists a probability distribution Q on B such that

$$\sum_{b\in B} q_b \cdot g(a,b) \geq \lambda \tag{3}$$

for all $a \in A$. Let I_1 be the set of λ such that there exists P satisfying (2) and let I_2 be the set of λ for which there exists Q such that (3) is satisfied. Then I_1 and I_2 are closed and $I_1 \cup I_2 = \mathbb{R}$. Thus I_1 and I_2 have non-empty intersection, and for λ in the intersection the inequalities (2) and (3) express that the pair (P,Q) is a Nash equilibrium and that λ is the value of the game. ■

The interpretation of the Dutch Book Theorem is that probability theory works as a safety net which prevent people from making incoherent decisions. the importance of the proof that the Dutch Book Theorem is equivalent to the existence of a Nash equilibrium for two-person zero-sum games is that it means that the two results refer to the same type of rationality.

In generally the probability distributions will not be uniquely determined, but will merely be located in a non-empty convex set. Therefore the Dutch Book Theorem suggests that uncertainty about some unknown event should be represented by a *convex set of probability distributions* rather than a single distribution.

We note that the Dutch book theorem tells nothing about how to determine probability distributions which will lead to good decisions. In a Bayesian approach to probability and statistics one will assign subjective probabilities expressing the individual feeling of how probable or likely an event is. Experiments have demonstrated that most people have a bad intuition of probabilities and are unable to assign probabilities in a consistent manner.

If either g is acceptable or $-g$ is acceptable then it is called a two-sided bet. In general people will find it difficult to decide that either g or $-g$ is acceptable and thus the two-sided bet is not realistic. In De Finetti [9] only two-sided bets were considered. In this case the convex set of probability distributions reduces to a point. It puts a greater demand on a person's ability to predict the outcome. In the above formulation of the Dutch Book Theorem we just have a one-sided bet with a set of acceptable payoff functions. The term one-sided bet is taken from F. Hampel [15], [16]. See [34] for a newer presentation of De Finetti's theorem.

A special case that has been studied in great detail is when the functions $g(\cdot,b)$ only assume two different values. In this special case the Dutch Book leads to the so-called *Dempster-Shafer belief functions* [11,12, 18,19,25–28,31].

It is worth noting that the theorem is understood more easily if a probability distribution P is identified with the mean value function E_P rather than the probability vector $(p_1, p_2, ...p_n)$ (see also [35]). For readers who are interested in quantum theory it will be a help because only the state/expectation is defined on a quantum system. Probabilities will only appear after a quantum measurement has been performed.

A serious problem related to the Dutch Book Theorem is that it relies on the concept of a payoff function that should be additive. One may think of the payoff function as money but it the uniform mean value of having 1.000.000 £ and having 0 £ is having 500.000 £, but most people have a very clear preference for having 500.000 £ rather than an unknown amount of money with mean value 500.000 £. Therefore one may think of the payoff as *value,* but this is also a highly debatable concept and one may consider money as an attempt to quantify value. Savage showed that the concept of value and payoff function can be replaced by the concept of preference, so that a coherent set of preferences corresponds to the existence of both a payoff function and a probability measure and this line of research has been followed up by may other researchers [6, 30]. Nevertheless the whole concept of value obscures the interpretation of what probabilities are. We will look for an alternative.

4 Combinatorial games

The theory of combinatorial games was developed by J. Conway as a tool to analyze board games [4, 7]. In a board game the players alternate in making moves. Each move changes the configuration of the pieces on the board to some other configuration but only certain changes are allowed. It is convenient to call the two players *Left* and *Right.* We shall often consider different board configurations as different games. If G denotes a game, i.e. a certain configuration then the game is specified by the configurations G^L that Left is allow to move to and the configurations G^R that Right is allowed to move to, and we write $G = \left\{G^L \mid G^R\right\}$. Now the point is that G^L and G^R are sets of games, so a game is formally a specification of two sets of games. In a board game it is nice for Left to have many options to choose among and bad if there are only few options. The worst case for Left is if there are *no options left* and in this case we say that Left has *lost* the game. So Left has lost the game if he is to move next and G^L is empty. Similar Right losses the game if it is Right to move and G^R is empty. The rules of many board games can be modelled in this way.

Example 7 *The game* $\{\varnothing \mid \varnothing\}$ *is a boring one. The one to move first loses this game. This game is denoted* 0.

The game $\{\varnothing \mid 0\}$ *is lost by Left if Left is to move first. If Right goes first Right has to choose* 0. *Now it is Left to move but this is a loosing position for the one who is going to move so poor Left loses. Thus Right always wins the game* $\{\varnothing \mid 0\}$. *This game is denoted* -1.

The game $\{0 \mid \varnothing\}$ *is lost by Right if Right is to move first. If Left goes first Left has to choose* 0. *Now it is Right to move but this is a loosing position for the one who is going to move so now Left is happy again because he wins. Thus Left always wins the game* $\{0 \mid \varnothing\}$. *This game is denoted* 1.

Similarly we see that $\{0 \mid 0\}$ *is won by the player that moves first. This game is denoted* $*$.

This calls for a recursive definition of a game.

Definition 8 *A game is a pair* $\left\{G^L \mid G^R\right\}$ *where* G^L *and* G^R *are sets of already defined games*

The status of a game G can be classified according to who wins if both players play optimally. We define

$$\begin{aligned} G &> 0, && \text{if Left wins whoever plays first;} \\ G &< 0, && \text{if Right wins whoever plays first;} \\ G &\| 0, && \text{if first player wins;} \\ G &= 0, && \text{if second player wins.} \end{aligned}$$

For a game G we can reverse the role of Left and Right and call this the negative of the game. Formally we use the following recursive definition.

$$-\left\{G^L \mid G^R\right\} = \left\{-G^R \mid -G^L\right\}.$$

Left and Right can play two game in parallel. In each round the players should make a move in one of the games of their own choice. Perhaps there are urgent moves to be made in both games so the players have to prioritize in which game it is most important to make the move. Several games played in parallel is called the sum of the games, and many positions in actual board games can be understood as sums of sub-games. Combinatorial game theory is essentially the theory of how to prioritizes your moves in a board game that has the structure of a sum of independent sub-games. Formally the sum of the games G and H is defined recursivelyby

$$\begin{aligned} &G + H = \\ &\left\{\left(G^L + H\right) \cup \left(G + H^L\right) \mid \left(G^R + H\right) \cup \left(G + H^R\right)\right\}. \end{aligned}$$

Now, we are able to define what it should mean that two games are equal. We write $G = H$ if $G - H = 0$, where $G-H$ is short for $G+(-H)$. One can define $G>H$, $G<H$, and $G\|H$ in the same way. We say that G and H are *confused* if $G\|H$. With these operations the class of games has the structure of a partially ordered group.

If n is a natural number we define $n\cdot G$ by

$$\overbrace{G+G+\cdots+G}^{n\text{ times}}.$$

The natural number n can now be identified with the game $n\cdot 1$, and a negative integer n can be identified with the game $-n\cdot -1$. Dyadic rationals (rational numbers of the form $n/2^m$) can also be identified with games. For instance the game $\{0\mid 1\}$ should be identified with $1/2$ because

$$\{0\mid 1\}+\{0\mid 1\}=1.$$

A game G is said to be *short* if it only has finitely many positions. Our recursive definition of games allow transfinite recursion and games that are not short, but in this section we shall focus on the short games.

Although games may be confused with a whole interval of numbers one can define the *mean value* $\mu(G)$ of a short game G. The mean value of a game is a real number that satisfies the following mean value theorem.

Theorem 9 ([7]) *If G is a short game then there exists a natural number m such that the mean value of G satisfies*

$$n\cdot\mu(G)-m\leq n\cdot G\leq n\cdot\mu(G)+m$$

for all natural numbers n.

Mean values of games behave much like mean values of random variables. For instance we have

- $\mu(n\cdot G)=n\cdot\mu(G)$,
- $\mu(G+H)=\mu(G)+\mu(H)$,
- $G\geq 0\Rightarrow\mu(G)\geq 0$,
- $\mu(1)=1$.

From Theorem 9 we see that the mean value as a function from games to numbers is uniquely determined by the the above conditions. Some game are positive but have mean value 0. Mean values of short games can be calculated by the thermographic method [7] and using this method it is easy to see that the mean value of a short game is a rational number.

Example 10 *The game $\{0\mid *\}$ is called* up *and denoted $\uparrow$. It is easy to check that $\uparrow\ >0$. The inequality $n\cdot\uparrow\ \leq 2$ is requires a little more strategic considerations. The inequalities lead to the inequality*

$$0\leq\mu(G),$$
$$n\mu(G)\leq 2,$$

for all natural numbers n. Hence, $\mu(G)=0$. The game $\uparrow$ is said to be all small.

5 Dutch books for combinatorial games

In the Dutch book theorem and in social games the payoff functions have real numbers as values. As J. Conway pointed out numbers can be considered as special types of games so an obvious generalization of payoff functions are functions where the values are games [7]. The setup is that each bookmaker tells Alice which game he wants to play if a certain horse wins. Alice is going to play Left and the bookmaker or the bookmakers are going to play Right. Some of these bookmakers are accepted and others are rejected depending on how favorable the games are. After certain bookmakers have been accepted the bookmakers choose natural numbers $n_b, b\in B$ and combine these into a super game $\sum_{b\in B}n_b\cdot G(a,b)$ that will depend on which horse wins. We say that we have a *Dutch book* if there exists natural numbers $n_1,n_2,\cdots,n_k$ such that Alice will lose the game

$$\sum_{b\in B}n_b\cdot G(a,b)\tag{4}$$

for any value of a. Otherwise the set of payoff functions is said to *coherent.*

You are then allowed to choose that the game G should be played a number of times in parallel. With this setup we get the following version of the Dutch book theorem.

Theorem 11 *If a set of payoff functions $G(a,b)$, $a\in A, b\in B$ with short games as values, is coherent then either exists a probability vector $a\to p_a$ and a natural number n such that $np_a\in\mathbb{N}$ and the game*

$$\sum_a(np_a)\cdot G(a,b)>0,\ \textit{for all } b\in B,\tag{5}$$

or there exist natural numbers $n_1,n_2,\cdots,n_k$, a natural number n and a probability vector $a\to p_a$ such that both games (4) and (5) have mean zero.

Proof. We apply the existence of an equilibrium in the social game with payoff function $(a,b)\to$

$\mu(G(a,b))$. If the value of the two-person sero-sum game is negative then the game (4) is negative if the coefficients $n_1, n_2, \cdots, n_k$ are large enough. If the value of the two-person zero-suml game is non-negative there exists a probability vector $a \to p_a$ such that

$$\sum_a p_a \cdot \mu(G(a,b)) \geq 0.$$

The mean value of a short game is a rational number. Therefore the probability vector $a \to p_a$ can be chosen with rational point probabilities. Therefore there exists a natural number m such that $m \cdot p_a$ is an integer for all $a \in A$. Hence

$$\begin{aligned} 0 &\leq m \sum_a p_a \cdot \mu(G(a,b)) \\ &\leq \sum_a m p_a \cdot \mu(G(a,b)) \\ &= \mu\left(\sum_a m p_a \cdot G(a,b)\right). \end{aligned}$$

If

$$\mu\left(\sum_a m p_a \cdot G(a,b)\right) > 0$$

then there exists a natural number k such that

$$k \sum_a m p_a \cdot G(a,b) > 0$$

and the game defined in (5) is winning for Alice who plays as Left when $n = km$. Otherwise

$$\mu\left(\sum_a m p_a \cdot G(a,b)\right) = 0.$$

■

The main difference between the ordinary Dutch book theorem and the version with a payoff function that has games as values lies in the existence of all small games. Even if none of the games $G(a,b)$ are all small, they may be combined into all small games. A simple example is the games 1 and $\uparrow -1$ whose sum is the all small game $\uparrow$.

The setup where bookmakers offer games may seem quite contrived, but many board games that involve chance are of this form.

Example 12 *In chess it is normally considered a slight advantage to play white. Therefore one normally randomly selects who should play white and who should play black.*

Example 13 *In handicap go the weakest player is allowed to start with a number of stones on the board determined by the difference in the strength of the players. Normally these handicap stones are placed at fixed locations but some players distribute the handicap stones randomly in order to create more variation in the game.*

Actually any board game involving chance may be considered as an example, but in most board games involving dice or similar randomness generators the randomness generator is used many times, which makes the analysis somewhat different in practice. It will be the subject of a future paper how to take advantage of a combined probabilistic and combinatorial game approach for some real world board games.

An interesting situation is when all games $G_b(a)$ are imfinitisimal. In this case the Dutch book theorem for games as formulated in Theorem 11 tells exactly nothing. But if all games are infinitesimal one can simply use a different "mean value" concept. For short games one compares the game with $n \cdot 1$ and the game 1 can be considered as a unit in the theory. For infinitesimal games one can compare with the infinitisimal game $\uparrow$ instead. This is called the *atomic weight theory* and works essentially in the same way as the ordinary mean value theory except that the proofs are more involved. Therefore one can also prove a Dutch book theorem for infinitesimally small games that involves three cases. The three cases are Dutch book, positive mean, and some *double infinitesimal games* G that cannot be analyzed by in the sense that $-m \cdot \uparrow \leq n \cdot G \leq m \cdot \uparrow$ for some natural number m and all natural numbers n.

We have seen that the Dutch Book theorem is equivalent to the existence of a Nash equilibrium in two-person zero-sum games. Therefore it is easy to formulate and prove a result on equilibria in combinatorial game valued two-person zero-sum games. For social games with more players and game valued payoff functions the situation is more complicated, and we shall not discuss it here as it has less interest for our understanding of what probabilities are.

6 Frequential Probabilities and Exchangeable Sequences

Let $X_1, X_2, \cdots$ be a sequence of Bernoulli random variables, i.e. only the values 0 and 1 are possible. We assume that our knowledge about them is given by a convex set K of probability measures. Let P be a probability measure on sequences and π be a permutation of $\{1, 2, \cdots, k\}$. Then the probability measure $\pi(P)$ given by

$$\pi(P)(X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k) = \\ P\left(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \cdots, X_k = x_{\pi(k)}\right). \quad (6)$$

Definition 14 *We say that (our knowledge of) the sequence $X_1, X_2, \cdots$ is weakly interchangeable if $P \in K$ implies $\pi(P) \in K$ for all permutations π. The sequence is strongly interchangeable if $P \in K$ implies that $P_\pi = P$.*

The concepts of weak and strong interchangeability are related as follows. Let K be a set of probability measures that gives weakly interchangeable sequence. For a set of permutations Π let K_Π denote the set of probability measures P such that $P = \pi(P)$ for $\pi \in \Pi$. Assume that K is compact, non-empty, and weak interchangeable. If $P \in K$ then $\frac{1}{|\pi|}\sum_{i=1}^{|\pi|}\pi^i(P) \in K$ and is invariant under π. Assume that K is compact and non-empty. Then K_Π is is compact if Π is a finite set of permutations. Hence, K_Π is a decreasing filter of non-empty compact sets and the intersection of all $\bigcap_\Pi K_\Pi$ is compact and non-empty. This set equals the set probability measures in K that are invariant under permutations. We also have that $\bigcap_\Pi K_\Pi$ is strongly interchangeable.

Example 15 *If $s \in [0;1]$ then the following formula gives a permutation invariant probability measure*

$$P(X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k) = \prod_{i=1}^{k} s^{x_i}(1-s)^{1-x_i}.$$

Theorem 16 *Let $X_1, X_2, \cdots$ be an infinite sequence of strongly exchangeable Bernoulli random variables. Then there exists a convex set C of probability distributions on $[0;1]$ so that $P \in K$ if and only if $P = P_\mu$ for some $\mu \in C$, where P_μ is defined by*

$$P_\mu(X_i = x_i, i \in D) = \int_0^1 \prod_{i \in D} s^{x_i}(1-s)^{1-x_i}\ d\mu s \quad (7)$$

for any finite subset $D \subset \mathbb{N}$.

Proof. According to de Finetti's Theorem [8, 10] $\mu \to P_\mu$ gives a one-to-one correspondence between probability measures on $[0;1]$ and probability measures on sequences of Bernoulli random variables such that (6) holds for all finite permutations. The mapping $\mu \to P_\mu$ is affine so the preimage of K is a convex and compact set. ■

The parameter s in Theorem 6 plays the role of the frequential probability. We see that the role played by frequential probabilities is exactly the same whether we consider uncertainty represented by a single probability measure or a convex set of probability measures as long as we use the concept of strong exchangeability. This result can also be generalized to other repetitive structures as defined by S. Lauritzen [22].

Like the Dutch Book theorem the idea of exchangeability was first formulated by F. Ramsay [29] who developed this idea into a high standard. It was about 10 years later that De Finetti independently reintroduced this concept, and although he was not the first on this subject he became much more influential, which seems primarily related to the fact that Ramsay died young whereas De Finetti continued to promote his ideas until a high age.

7 Surreal probabilities

As we have seen all integers can be considered as special types of games. Also some non-integers have been identified with games. J. Conway discovered that all real numbers can be identified with games but his construction will lead to a larger class of numbers called the *surreal numbers* (or Conway numbers). The surreal numbers were first described in a mathematical novel by D. Knuth [20], and later in much detail by J. Conway [7]. For a newer and more complete description we refer to [23] and [14].

The construction is recursive. We start with the number $0 = \{|\}$. In each recursion step one adds new surreal numbers to the ones already constructed. The new numbers are all Dedekind sections in the set of already constructed surreal numbers. One continues via transfinite induction. Any integer or dyadic rational number can be constructed in a finite number of recursion steps. All real numbers that are not dyadic ratianl numbers, are constructed in the first transfinite step, but the recursion does not stop there. The surreal numbers contain many more numbers than the real numbers. Between any two different real numbers the are more than continuously many surreal numbers, and the intersection of the intervals $[-1/n; 1/n]$ contains an infinity of *infinitesimals*. There are also many surreal numbers that are greater than any integer and many surreal numbers that are less than any integer. Addition and multiplication extend to surreal numbers and with these operations the surreal numbers is a maximal ordered field. For most computations they are not different from ordinary real numbers but obviously the topology is different. Formally the surreal numbers do not form a set but a class. A Dutch book for a set of surreal payoff functions $g(a,b), a \in A, b \in B$ is a set of non-negative surreal numbers q_b such that $\sum q_b = 1$ and

$$a \to \sum_{b\in B} q_b \cdot g(a,b) \tag{8}$$

is negative for all $a \in A$. If no Dutch book exists the set of payoff functions is said to be coherent.

Because of the somewhat different topology of the surreal numbers, we to be a little careful in the Dutch book theorem and the equivalent existence of Nash equilibrium two-persons zero-sum games. In particular some of the standard methods for proving these results like Hahn-Banach theorem and the separation theorem for convex sets, do not hold in their normal formulation when we are using surreal numbers.

Theorem 17 *Let A and B denote finite sets and let $(a,b) \to g(a,b)$ denote a surreal payoff function. If the payoff function is coherent then there exists non-negative surreal numbers p_a such that $\sum p_a = 1$ and*

$$\sum_{a\in A} p_a \cdot g(a,b) \geq 0$$

for all $b \in B$..

Proof. Assume that A has d elements. Then each function $g(\cdot,b)$ may be identified with a point in $\mathbb{R}_s^d$. Let K be the convex hull of $\{g(\cdot,b) \mid b \in B\}$, and let $L_-^\infty(A)$ denote the strictly negative surreal functions on A. They are convex classes.

If K and $L_-^\infty(A)$ intersect then there exists non-negative surreal numbers q_b such that $\sum q_b = 1$ and such that (8) defines a strictly negative function.

Assume that K and $L_-^\infty(A)$ are disjoint. Then define $C = K - L_-^\infty(A)$ as the class of vectors $\bar{x}-\bar{y}$ where $\bar{x}$ in K and $\bar{y}$ in $L_-^\infty(A)$.This is convex and does not contain $\bar{0}$.Now, K is a polytypic (convex hull of finitely many extreme points) and $L_-^\infty(A)$ is polyhedral (given by finitely many inequalities), so C is polyhedral. Hence, each of the faces of C is given by a linear inequality of the form $\sum_{a\in A} p_a \cdot g(a) \geq c$ for $g \in C$. The delta function δ_α is non-negative so is g is in C then $g - \ell \cdot \delta_b$ is also in C for ℓ positive. In particular

$$\begin{aligned} c &\leq \sum_{a\in A} p_a \cdot (g - \ell \cdot \delta_\alpha)(a) \\ &= \sum_{a\in A} p_a \cdot g(a) - \sum_{a\in A} p_a \ell \delta_\alpha(a) \\ &= \sum_{a\in A} p_a \cdot g(a) - \ell \cdot p_\alpha \end{aligned}$$

for all positive ℓ. Hence $p_\alpha \geq 0$ for all $\alpha \in A$. Further we know that $\bar{0}$ is not in C so that $\sum_{a\in A} p_a \cdot 0 \geq c$ does not hold and therefore $c > 0$. In particular p_a cannot be 0 for all a. The result follows by replacing p_a by

$$\frac{p_a}{\sum_{a\in A} p_a}.$$

■

The theorem leads to surreal probabilities p_a that satisfy $0 \leq p_a \leq 1$. Although probabilities are surreal they cannot be infinite, but there is no problem in having infinitesimal probabilities. In any frequential interpretation of probability theory, probabilities should be interpreted as limits of frequencies. Obviously surreal probabilities cannot have such interpretations because a frequential interpretation cannot distinguish between surreal probabilities that have a infinitesimal difference.

According to Kolmogorov [21] conditional probabilities can be considered as a derived concept given by the formula

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

This point of view has been criticized for several reasons. One is that many feel that conditional probabilities should be a more fundamental concept because we often have a clear idea about the value of conditional probabilities but no clear idea about the joint probabilities. Another related problem is that one sometimes want to condition on an event that has probability 0. If one allows surreal probabilities we note that the formula for conditional probabilities still works if $P(B)$ is infinitesimal. The problem of conditioning on a set of measure 0 most often happen when one works with continuous distributions, so in order to discuss this problem in more depth one would have to develop measure theory where the measure of a set is allowed to have a surreal value, and cannot be done in this short paper.

8 Discussion

In this paper we have identified the coherence with the non-existence of Dutch books. This definition is based on the classical Dutch Book Theorem. We have seen that if the payoff function is a game then the situation is somewhat more complicated because of the existence of all small positive or negative games. One obvious way to handle this situation would be to change the definition of coherence, but this would of cause not solve the problem that games are more complicated objects than numbers.

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