# On the Number of Hamiltonian Groups

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**Abstract**

Finite hamiltonian groups are counted. The sequence of numbers of all groups of order $n$ all whose subgroups are normal and the sequence of numbers of all groups of order less or equal to $n$ all whose subgroups are normal are presented.

**Keywords:** group, number, sequence, normal subgroup, abelian, hamiltonian.
**MSC 2000:** 11Y55, 05C25, 20B05.

## 1 Introduction

Subgroups of abelian groups are abelian and hence self-conjugate or *normal*. A nonabelian group all of whose subgroups are normal is called *hamiltonian* [1, 14]. Let $\mathcal{A}$ denote the class of abelian groups and let $\mathcal{H}$ denote the class of hamiltonian groups. In topological graph theory [2, 15], hamiltonian groups have been studied in the past [5, 7, 6]. For several classes of hamiltonian groups the genus is known exactly. For abelian and hamiltonian groups, there are structural theorems available. We note in passing that here we use a different structure theorem. For instance, the cyclic group $\mathbb{Z}_{15}$ can be written as $\mathbb{Z}_3 \times \mathbb{Z}_5$. Since it can be generated by a single generator, the former form is preferred in the topological graph theory over the latter. In this paper we determine the number $h(n)$ of hamiltonian groups of order $n$ and the number $b(n)$ of all groups of order $n$ with the property, that all their subgroups are normal. We also determine the number $v(n)$ of all hamiltonian groups of order $\leq n$ and the number $w(n)$ of all groups of order $\leq n$ with the property, that all their subgroups are normal.

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## 2 Results

Before we study hamiltonian groups we will recall the structure of finite abelian groups [13]. Let $\pi(m)$ denote a partition of a natural number $m$, where

$$\pi(m) := \{m_1, m_2, \ldots, m_s\},$$

such that $m = \sum_{k=1}^{s} m_k$ and $m_i \geq m_j$ for all $1 \leq i < j \leq s$. For $c \in \mathbb{N}$ let $c^{\pi(m)} := \{c^{m_1}, c^{m_2}, \ldots, c^{m_s}\}$ and let $A(n_1, n_2, \ldots, n_r)$ denote the direct product of cyclic groups

$$A(n_1, n_2, \ldots, n_r) := \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}.$$

Let $G$ be a finite abelian group of order $n$. Let us write down the prime decomposition of $n$ as

$$n = \prod_{k=1}^{\ell} p_k^{\alpha_k}.$$

It is well-known that $G$ is isomorphic to the group

$$G \approx A\left(p_1^{\pi(\alpha_1)}, p_2^{\pi(\alpha_2)}, \ldots, p_\ell^{\pi(\alpha_\ell)}\right).$$

Let $a(n)$ denote the number of abelian groups of order $n$ and let $P(n)$ denote the number of partitions of the integer $n$. The previous discussion gives a proof to the following result.

**Proposition 1.** *The number $a(n)$ of abelian groups of order $n$ is given by $\prod_{i=1}^{\ell} P(\alpha_i)$ where $n = \prod_{k=1}^{\ell} p_k^{\alpha_k}$ is the prime decomposition of $n$.*

The initial 200 values of the sequence $a(n)$ are given in Table 1.

| $n$ | 1 | | | | 5 | | | | | 10 | | | | | 15 | | | | | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 5 | 1 | 2 | 1 | 2 |
| 20 | 1 | 1 | 1 | 3 | 2 | 1 | 3 | 2 | 1 | 1 | 1 | 7 | 1 | 1 | 1 | 4 | 1 | 1 | 1 | 3 |
| 40 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 5 | 2 | 2 | 1 | 2 | 1 | 3 | 1 | 3 | 1 | 1 | 1 | 2 |
| 60 | 1 | 1 | 2 | 11 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 6 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 5 |
| 80 | 5 | 1 | 1 | 2 | 1 | 1 | 1 | 3 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 7 | 1 | 2 | 2 | 4 |
| 100 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 6 | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 3 |
| 120 | 2 | 1 | 1 | 2 | 3 | 2 | 1 | 15 | 1 | 1 | 1 | 2 | 1 | 1 | 3 | 3 | 1 | 1 | 1 | 2 |
| 140 | 1 | 1 | 1 | 10 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 3 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 7 |
| 160 | 1 | 5 | 1 | 2 | 1 | 1 | 1 | 3 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 5 | 1 | 1 | 1 | 4 |
| 180 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 2 | 3 | 1 | 1 | 11 | 1 | 1 | 1 | 4 | 1 | 2 | 1 | 6 |

Table 1: The initial values of $a(n)$, $n = 1, 2, \ldots, 200$, ([8], A000688).

Note that the sequence $\{a(n)\}_{n\in\mathbb{N}}$ can not contain multiples of primes in the sequence $s := \{13, 17, 19, 23, 29, 31, 37, \ldots\}$ since $P(n) \neq k \cdot s_i, \forall n, i, k \in \mathbb{N}$, (see [9]). The number $a(n)$ depends only on the prime signature of $n$. For example, both $24 = 2^3 \cdot 3^1$ and $375 = 3^1 \cdot 5^3$ have the prime signature $(3, 1)$, therefore $a(375) = a(24)$.

A similar structural theorem holds for hamiltonian groups. A hamiltonian group $H$ is isomorphic to a direct product of the quaternion group $Q$ of order 8, an elementary abelian group $E$ of exponent 2 and an abelian group $A$ of odd order

$$H \approx Q \times E \times A \approx Q \times \mathbb{Z}_{2^k} \times A,$$

where $|Q| = 8 = 2^3$, $|E| = 2^k$ and $|A| \neq 0 \pmod 2$. Therefore $|H| = 2^{3+k}|A|$. Let $n$ be an arbitrary natural number. We can write $n$ uniquely in the form $n = 2^e \cdot o$ where $e = e(n) \geq 0$ and $o = o(n)$ is an odd number. These results give the number of hamiltonian groups of order $n$.

**Proposition 2.** *Let $n = 2^e \cdot o$, where $e = e(n) \geq 0$ and $o = o(n)$ is an odd number. The number $h(n)$ of hamiltonian groups of order $n$ is given by*

$$h(n) = \begin{cases} 0, & e(n) < 3; \\ a\left(o(n)\right), & \text{otherwise.} \end{cases}$$

The initial 200 values of the sequence $h(n)$ are given in Table 2.

| $n$ | 1 | | | | 5 | | | | | 10 | | | | | 15 | | | | | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 60 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 80 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 100 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 120 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 140 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 160 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 180 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |

Table 2: The initial values of $h(n)$, $n = 1, 2, \ldots, 200$.

Combining abelian and hamiltonian groups of order $n$ we may now give the number $b(n) := a(n) + h(n)$ of all groups of order $n$ all of whose subgroups are normal. The initial 300 values of the sequence $b(n)$ are given in Table 3.

| $n$ | 1 | | | | 5 | | | | | 10 | | | | | 15 | | | | | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 4 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 6 | 1 | 2 | 1 | 2 |
| 20 | 1 | 1 | 1 | 4 | 2 | 1 | 3 | 2 | 1 | 1 | 1 | 8 | 1 | 1 | 1 | 4 | 1 | 1 | 1 | 4 |
| 40 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 6 | 2 | 2 | 1 | 2 | 1 | 3 | 1 | 4 | 1 | 1 | 1 | 2 |
| 60 | 1 | 1 | 2 | 12 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 8 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 6 |
| 80 | 5 | 1 | 1 | 2 | 1 | 1 | 1 | 4 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 8 | 1 | 2 | 2 | 4 |
| 100 | 1 | 1 | 1 | 4 | 1 | 1 | 1 | 6 | 1 | 1 | 1 | 6 | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 4 |
| 120 | 2 | 1 | 1 | 2 | 3 | 2 | 1 | 16 | 1 | 1 | 1 | 2 | 1 | 1 | 3 | 4 | 1 | 1 | 1 | 2 |
| 140 | 1 | 1 | 1 | 12 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 4 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 8 |
| 160 | 1 | 5 | 1 | 2 | 1 | 1 | 1 | 4 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 6 | 1 | 1 | 1 | 4 |
| 180 | 1 | 1 | 1 | 4 | 1 | 1 | 1 | 2 | 3 | 1 | 1 | 12 | 1 | 1 | 1 | 4 | 1 | 2 | 1 | 8 |
| 200 | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 6 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 12 | 1 | 1 | 1 | 2 |
| 220 | 1 | 1 | 1 | 8 | 4 | 1 | 1 | 2 | 1 | 1 | 1 | 4 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 6 |
| 240 | 1 | 2 | 7 | 2 | 2 | 1 | 1 | 4 | 1 | 3 | 1 | 4 | 1 | 1 | 1 | 23 | 1 | 1 | 1 | 2 |
| 260 | 2 | 1 | 1 | 4 | 1 | 1 | 1 | 2 | 1 | 3 | 1 | 6 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 4 |
| 280 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 16 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 4 | 3 | 1 | 1 | 4 |

Table 3: The initial values of $b(n)$, $n = 1, 2, \ldots, 300$.

The number $u(n)$ of all abelian groups of order $\leq n$ is presented in [11]. The initial 100 values of the sequence $u(n)$ are given in Table 4.

| $n$ | 1 | | | | 5 | | | | | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 11 | 13 | 14 |
| 10 | 15 | 17 | 18 | 19 | 20 | 25 | 26 | 28 | 29 | 31 |
| 20 | 32 | 33 | 34 | 37 | 39 | 40 | 43 | 45 | 46 | 47 |
| 30 | 48 | 55 | 56 | 57 | 58 | 62 | 63 | 64 | 65 | 68 |
| 40 | 69 | 70 | 71 | 73 | 75 | 76 | 77 | 82 | 84 | 86 |
| 50 | 87 | 89 | 90 | 93 | 94 | 97 | 98 | 99 | 100 | 102 |
| 60 | 103 | 104 | 106 | 117 | 118 | 119 | 120 | 122 | 123 | 124 |
| 70 | 125 | 131 | 132 | 133 | 135 | 137 | 138 | 139 | 140 | 145 |
| 80 | 150 | 151 | 152 | 154 | 155 | 156 | 157 | 160 | 161 | 163 |
| 90 | 164 | 166 | 167 | 168 | 169 | 176 | 177 | 179 | 181 | 185 |

Table 4: The initial values of $u(n)$, $n = 1, 2, \ldots, 100$, ([11], A063966).

Let $v(n)$ be the number of all hamiltonian groups of order $\leq n$ and let $w(n)$ be the number of all groups of order $\leq n$ all of whose subgroups are normal. The initial 200 values of the sequences $v(n)$ and $w(n)$ are given in Table 5 and Table 6, respectively.

| $n$ | 1 | | | | 5 | | | | | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 |
| 20 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 30 | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 40 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 6 |
| 50 | 6 | 6 | 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 |
| 60 | 7 | 7 | 7 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 70 | 8 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 11 |
| 80 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 12 | 12 | 12 |
| 90 | 12 | 12 | 12 | 12 | 12 | 13 | 13 | 13 | 13 | 13 |
| 100 | 13 | 13 | 13 | 14 | 14 | 14 | 14 | 14 | 14 | 14 |
| 110 | 14 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 16 |
| 120 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 17 | 17 |
| 130 | 17 | 17 | 17 | 17 | 17 | 18 | 18 | 18 | 18 | 18 |
| 140 | 18 | 18 | 18 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
| 150 | 20 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 22 |
| 160 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 23 | 23 | 23 |
| 170 | 23 | 23 | 23 | 23 | 23 | 24 | 24 | 24 | 24 | 24 |
| 180 | 24 | 24 | 24 | 25 | 25 | 25 | 25 | 25 | 25 | 25 |
| 190 | 25 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 28 |

Table 5: The initial values of $v(n)$, $n = 1, 2, \ldots, 200$.

| $n$ | 1 | | | | 5 | | | | | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 12 | 14 | 15 |
| 10 | 16 | 18 | 19 | 20 | 21 | 27 | 28 | 30 | 31 | 33 |
| 20 | 34 | 35 | 36 | 40 | 42 | 43 | 46 | 48 | 49 | 50 |
| 30 | 51 | 59 | 60 | 61 | 62 | 66 | 67 | 68 | 69 | 73 |
| 40 | 74 | 75 | 76 | 78 | 80 | 81 | 82 | 88 | 90 | 92 |
| 50 | 93 | 95 | 96 | 99 | 100 | 104 | 105 | 106 | 107 | 109 |
| 60 | 110 | 111 | 113 | 125 | 126 | 127 | 128 | 130 | 131 | 132 |
| 70 | 133 | 141 | 142 | 143 | 145 | 147 | 148 | 149 | 150 | 156 |
| 80 | 161 | 162 | 163 | 165 | 166 | 167 | 168 | 172 | 173 | 175 |
| 90 | 176 | 178 | 179 | 180 | 181 | 189 | 190 | 192 | 194 | 198 |
| 100 | 199 | 200 | 201 | 205 | 206 | 207 | 208 | 214 | 215 | 216 |
| 110 | 217 | 223 | 224 | 225 | 226 | 228 | 230 | 231 | 232 | 236 |
| 120 | 238 | 239 | 240 | 242 | 245 | 247 | 248 | 264 | 265 | 266 |
| 130 | 267 | 269 | 270 | 271 | 274 | 278 | 279 | 280 | 281 | 283 |
| 140 | 284 | 285 | 286 | 298 | 299 | 300 | 302 | 304 | 305 | 307 |
| 150 | 308 | 312 | 314 | 315 | 316 | 318 | 319 | 320 | 321 | 329 |
| 160 | 330 | 335 | 336 | 338 | 339 | 340 | 341 | 345 | 347 | 348 |
| 170 | 350 | 352 | 353 | 354 | 356 | 362 | 363 | 364 | 365 | 369 |
| 180 | 370 | 371 | 372 | 376 | 377 | 378 | 379 | 381 | 384 | 385 |
| 190 | 386 | 398 | 399 | 400 | 401 | 405 | 406 | 408 | 409 | 417 |

Table 6: The initial values of $w(n)$, $n = 1, 2, \ldots, 200$.

If we look at the sequences $\{a(n)\}_{n\in\mathbb{N}}$ and $\{h(n)\}_{n\in\mathbb{N}}$ from the inverse perspective, we can define two more sequences. Let $S_a(n)$ denote the smallest number $k \in \mathbb{N}$, for which exactly $n$ nonisomorphic abelian groups of order $k$ exist ([10]). The first 60 elements of the sequence $\{S_a(n)\}_{n\in\mathbb{N}}$ are given in Table 7. Here 0 denotes the case, where $S_a(n)$ does not exist ($n$ is not a product of partition numbers). These indices $n$ are exactly multiples of primes in the sequence $s$ ([9]).

| $n$ | 1 | | | | 5 | | | | | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 8 | 36 | 16 | 72 | 32 | 900 | 216 | 144 |
| 10 | 64 | 1800 | 0 | 288 | 128 | 44100 | 0 | 5400 | 0 | 3600 |
| 20 | 864 | 256 | 0 | 88200 | 1296 | 0 | 27000 | 7200 | 0 | 512 |
| 30 | 0 | 5336100 | 1728 | 0 | 2592 | 264600 | 0 | 0 | 0 | 176400 |
| 40 | 0 | 1024 | 0 | 2304 | 3456 | 0 | 0 | 10672200 | 7776 | 32400 |
| 50 | 0 | 0 | 0 | 1323000 | 5184 | 2048 | 0 | 0 | 0 | 4608 |

Table 7: The initial values of $S_a(n)$, $n = 1, 2, \ldots, 60$, ([10], A046056).

Let $S_h(n)$ denote the smallest number $k \in \mathbb{N}$, for which exactly $n$ nonisomorphic hamiltonian groups of order $k$ exist. The first 30 elements of the sequence $\{S_h(n)\}_{n\in\mathbb{N}}$ are given in Table 8, where again 0 denotes the case, where $n$ is not a product of partition numbers and $S_h(n)$ does not exist.

| $n$ | 1 | | | | 5 | | | | | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 8 | 72 | 216 | 1800 | 648 | 5400 | 1944 | 88200 | 27000 | 16200 |
| 10 | 5832 | 264600 | 0 | 48600 | 17496 | 10672200 | 0 | 1323000 | 0 | 793800 |
| 20 | 243000 | 52488 | 0 | 32016600 | 405000 | 0 | 9261000 | 2381400 | 0 | 157464 |

Table 8: The initial values of $S_h(n)$, $n = 1, 2, \ldots, 30$.

Let us finish with two open problems. Think of computing the genus of each of the groups $\Gamma \in \mathcal{A} \cup \mathcal{H}$, counted by $b(n)$. Since $\mathbb{Z}_n \in \mathcal{A} \cup \mathcal{H}$, the minimal genus is 0. A natural question is therefore to determine

$$g(n) := \max\{\gamma(\Gamma) \mid \Gamma \in \mathcal{A} \cup \mathcal{H},\ |\Gamma| = n\}.$$

The sequence $\{g(n)\}_{n\in\mathbb{N}} = (0, 0, 0, 0, 0, 0, 0, 1, \ldots)$ .

Another interesting problem is a generalization of the considered problem, namely, the problem of determining the number of groups, whose every subgroup is 2-subnormal ([4, 12]). A subgroup $H$ of group $G$ is said to be *2-subnormal* in $G$ if there is a series

$$H = H_2 \lhd H_1 \lhd H_0 = G$$

of subgroups in $G$ (see [3]). Such a subgroup is said also to be *of defect* 2. Similarly, subgroups $H$ of defect 1 in $G$ are precisely normal subgroups of $G$.

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