

Complexity of the Minimum Cost Homomorphism Problem for Semicomplete Digraphs with Possible Loops

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Abstract

For digraphs D and H , a mapping $f : V(D) \rightarrow V(H)$ is a homomorphism of D to H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. For a fixed digraph H , the homomorphism problem is to decide whether an input digraph D admits a homomorphism to H or not, and is denoted as $\text{HOM}(H)$.

An optimization version of the homomorphism problem was motivated by a real-world problem in defence logistics and was introduced in [12]. If each vertex $u \in V(D)$ is associated with costs $c_i(u), i \in V(H)$, then the cost of the homomorphism f is $\sum_{u \in V(D)} c_{f(u)}(u)$. For each fixed digraph H , we have the *minimum cost homomorphism problem for H* and denote it as $\text{MinHOM}(H)$. The problem is to decide, for an input graph D with costs $c_i(u), u \in V(D), i \in V(H)$, whether there exists a homomorphism of D to H and, if one exists, to find one of minimum cost.

Although a complete dichotomy classification of the complexity of $\text{MinHOM}(H)$ for a digraph H remains an unsolved problem, complete dichotomy classifications for $\text{MinHOM}(H)$ were proved when H is a semicomplete digraph [9], and a semicomplete multipartite digraph [11, 10]. In these studies, it is assumed that the digraph H is loopless. In this paper, we present a full dichotomy classification for semicomplete digraphs with possible loops, which solves a problem in [8].

1 Introduction, Terminology and Notation

For directed (undirected) graphs G and H , a mapping $f : V(G) \rightarrow V(H)$ is a *homomorphism of G to H* if uv is an arc (edge) implies that $f(u)f(v)$ is an arc (edge). A homomorphism f of G to H is also called an *H -coloring* of G , and $f(x)$ is called the *color* of the vertex x in G . We denote the set of all homomorphisms from G to H by $\text{HOM}(G, H)$.

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Let H be a fixed directed or undirected graph. The *homomorphism problem*, $\text{HOM}(H)$, for H asks whether a directed or undirected input graph G admits a homomorphism to H . The *list homomorphism problem*, $\text{ListHOM}(H)$, for H asks whether a directed or undirected input graph G with lists (sets) $L_u \subseteq V(H)$, admits a homomorphism f to H in which $f(u) \in L_u$ for each $u \in V(G)$.

Suppose G and H are directed (or undirected) graphs, and $c_i(u)$, $u \in V(G)$, $i \in V(H)$ are nonnegative *costs*. The *cost of a homomorphism* f of G to H is $\sum_{u \in V(G)} c_{f(u)}(u)$. If H is fixed, the *minimum cost homomorphism problem*, $\text{MinHOM}(H)$, for H is the following optimization problem. Given an input graph G , together with costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, find a minimum cost homomorphism of G to H , or state that none exists.

The minimum cost homomorphism problem was introduced in [12], where it was motivated by a real-world problem in defence logistics. We believe it offers a practical and natural model for optimization of weighted homomorphisms. The problem's special cases include the list homomorphism problem [14, 17] and the general optimum cost chromatic partition problem, which has been intensively studied [13, 18, 19], and has a number of applications [20, 22].

If a directed (undirected) graph G has no loops, we call G *loopless*. If a directed (undirected) graph G has a loop at every vertex, we call G *reflexive*. When we wish to stress that a family of digraphs may contain digraphs with loops, we will speak of digraphs *with possible loops (w.p.l.)* For an undirected graph H , $V(H)$ and $E(H)$ denote its vertex and edge sets, respectively. For a digraph H , $V(H)$ and $A(H)$ denote its vertex and arc sets, respectively.

In this paper, we give a complete dichotomy classification of the complexity of $\text{MinHOM}(H)$ when H is a semicomplete digraph with possible loops. A dichotomy of $\text{MinHOM}(H)$ when H is a tournament w.p.l. was established in [8], but it is much easier than the more general dichotomy obtained in this paper. A full dichotomy of $\text{MinHOM}(H)$ for H being a (general) digraph has not been settled yet and is considered to be a very difficult open problem. Nonetheless, dichotomy have been obtained for special classes of digraphs such as semicomplete digraphs and semicomplete multipartite digraphs; see [9, 10, 11]. Note that, for a semicomplete digraph H , the dichotomy for $\text{MinHOM}(H)$ [9] is different from the dichotomy for $\text{HOM}(H)$ [2] and $\text{ListHOM}(H)$ [9] (the last two coincide). Our dichotomy for $\text{MinHOM}(H)$ when H is a semicomplete digraph w.p.l. is significantly more involved than the one for $\text{MinHOM}(H)$ when H is a (loopless) semicomplete digraph.

Notice that, apart from [8], all these studies deal only with loopless digraphs. When we study the structure of a digraph D , we usually assume that D has no loops. This is often a natural assumption since many properties of loopless digraphs can readily be extended to general digraphs w.p.l. as the loops do not affect important parts of the structure of a digraph in the majority of cases. When we investigate homomorphisms of undirected/directed graphs, the situation is different and loops have to be taken into

consideration in the general case.

The homomorphism problem $\text{HOM}(H)$ is trivially polynomial time solvable when H has a loop, since we may simply map all the vertices of the input graph to a vertex with a loop. However, if we wish to get a dichotomy of $\text{MinHOM}(H)$ or $\text{ListHOM}(H)$, it is not that simple. For example, in [7], it turns out that the class of proper interval graphs is exactly the class of reflexive graphs for which $\text{MinHOM}(H)$ is polynomial time solvable (assuming, as usual, that $P \neq NP$). On the other hand, if we assume that H is loopless, $\text{MinHOM}(H)$ is polynomial time solvable if and only if H is a proper interval bigraph. It is often the case that, even if we succeeded in obtaining a dichotomy classification of $\text{MinHOM}(H)$ for reflexive and loopless H separately, it is another issue to get a dichotomy classification for H with possible loops.

Complete dichotomy classifications of $\text{ListHOM}(H)$ and $\text{MinHOM}(H)$, for an undirected graph H w.p.l., have been achieved, see [3, 4, 5] and [7]. For a directed graph with possible loops, the study has just begun and there are only a few results proved [8] so far. In [6], the authors prove some partial results on complexity of $\text{ListHOM}(H)$ when H is a reflexive digraph. In particular, it is conjectured that for a reflexive digraph H , $\text{ListHOM}(H)$ is polynomial time solvable if and only if H has a proper ordering. Here, we say that a reflexive digraph H has a *proper ordering* if its vertices can be ordered so that whenever $xy, x'y' \in A(H)$, $\min(x, x') \min(y, y')$ is also in $A(H)$. Unfortunately, the conjecture remains unconfirmed even for the case of reflexive semicomplete digraphs.

In the rest of this section, we give additional terminology and notation. In the subsequent sections, we first prove a full dichotomy classification of the complexity of $\text{MinHOM}(H)$ when H is a reflexive semicomplete digraph. Using this result, we will further present a full dichotomy classification of $\text{MinHOM}(H)$ when H is a semicomplete digraph with possible loops.

For a digraph D , if $xy \in A(D)$, we say that x *dominates* y and y *is dominated* by x , denoted by $x \rightarrow y$. Furthermore, if $xy \in A(D)$ and $yx \notin A(D)$, then we say that x *strictly dominates* y and y *is strictly dominated* by x , denoted by $x \mapsto y$. For sets $X, Y \subseteq V(G)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X, y \in Y$. Also, for sets $X, Y \subseteq V(G)$, $X \mapsto Y$ means that $xy \in A(D)$ but $yx \notin A(D)$ for each $x \in X, y \in Y$. For $xy \in A(D)$, we call xy an *asymmetric arc* if $yx \notin A(D)$, and a *symmetric arc* if $yx \in A(D)$. A digraph D is *symmetric* if each arc of D is symmetric. For a digraph H , H^{sym} denotes the *symmetric subdigraph* of H , i.e., a digraph with $V(H^{sym}) = V(H)$ and $A(H^{sym}) = \{uv, vu \in A(H)\}$. Note that any vertex u of $V(H^{sym})$ has a loop if and only if u has a loop in H . We call a directed graph D an *oriented graph* if all arcs of D are asymmetric.

For a digraph D , let $D[X]$ denote a subdigraph induced by $X \subseteq V(D)$. For any pair of vertices of a directed graph D , we say that u and v are *adjacent* if $u \rightarrow v$ or $v \rightarrow u$, or both. The *underlying graph* $U(D)$ of a directed graph D is the undirected graph obtained from D by disregarding all orientations and deleting one edge in each pair of parallel

edges. A directed graph D is *connected* if $U(D)$ is connected. The *components* of D are the subdigraphs of D induced by the vertices of components of $U(D)$.

By a *directed path (cycle)* we mean a simple directed path (cycle) (i.e., with no self-crossing). We assume that a directed cycle has at least two vertices. In particular, a loop is not a cycle. A directed cycle with k vertices is called a *directed k -cycle* and denoted by \vec{C}_k . Let K_n^* denote a complete digraph with a loop at each vertex, i.e., a reflexive complete digraph.

An *empty digraph* is a digraph with no arcs. A loopless digraph D is a *tournament (semicomplete digraph)* if there is exactly one arc (at least one arc) between every pair of vertices. We will consider *semicomplete digraphs with possible loops (w.p.l.)*, i.e., digraphs obtained from semicomplete digraphs by appending some number of loops (possibly zero loops). A *k -partite tournaments (semicomplete k -partite digraph)* is a digraph obtained from a complete k -partite graph by replacing every edge xy with one of the two arcs xy, yx (with at least one of the arcs xy, yx). An acyclic tournament on p vertices is denoted by TT_p and called a *transitive tournament*. The vertices of a transitive tournament TT_p can be labeled $1, 2, \dots, p$ such that $ij \in A(TT_p)$ if and only if $1 \leq i < j \leq p$. By TT_p^- ($p \geq 2$), we denote TT_p without the arc $1p$. For an acyclic digraph H , an ordering u_1, u_2, \dots, u_p is called *acyclic* if $u_i \rightarrow u_j$ implies $i < j$.

Let H be a digraph. The *converse* of H is the digraph obtained from H by replacing every arc xy with the arc yx . For a pair X, Y of vertex sets of a digraph H , we define $X \times Y = \{xy : x \in X, y \in Y\}$. Let H be a loopless digraph with vertices x_1, x_2, \dots, x_p and let S_1, S_2, \dots, S_p be digraphs. Then the *composition* $H[S_1, S_2, \dots, S_p]$ is the digraph obtained from H by replacing x_i with S_i for each $i = 1, 2, \dots, p$. In other words,

$$V(H[S_1, S_2, \dots, S_p]) = V(S_1) \cup V(S_2) \cup \dots \cup V(S_p) \text{ and}$$

$$A(H[S_1, S_2, \dots, S_p]) = \cup\{V(S_i) \times V(S_j) : x_i x_j \in A(H), 1 \leq i \neq j \leq p\} \cup (\cup_{i=1}^p A(S_i)).$$

If every S_i is an empty digraph, the composition $H[S_1, S_2, \dots, S_p]$ is called an *extension* of H .

The *intersection graph* of a family $F = \{S_1, S_2, \dots, S_n\}$ of sets is the graph G with $V(G) = F$ in which S_i and S_j are adjacent if and only if $S_i \cap S_j \neq \emptyset$. Note that by this definition, each intersection graph is reflexive. A graph isomorphic to the intersection graph of a family of intervals on the real line is called an *interval graph*. If the intervals can be chosen to be inclusion-free, the graph is called a *proper interval graph*.

2 Classification for Reflexive Semicomplete Digraphs

In this section, we describe a dichotomy classification of the complexity of $\text{MinHOM}(H)$ when H is a reflexive semicomplete digraph. Let R be a reflexive digraph with $V(R) =$

$\{1, 2, 3\}$ and $A(R) = \{12, 23, 31, 13, 11, 22, 33\}$. Let \vec{C}_3^* denote a reflexive directed cycle on three vertices. The main dichotomy classification of this section is given in the following theorem.

Theorem 2.1 *Let H be a reflexive semicomplete digraph. If H does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(H^{sym})$ is a proper interval graph (possibly with more than one component), then $MinHOM(H)$ is polynomial time solvable. Otherwise, $MinHOM(H)$ is NP-hard.*

2.1 NP-hard cases of $MinHOM(H)$

The following lemma is an obvious basic observation often used to obtain dichotomies. This lemma is certainly applicable for a digraph H w.p.l.

Lemma 2.2 [9] *Let H' be an induced subdigraph of a digraph H . If $MinHOMP(H')$ is NP-hard, then $MinHOMP(H)$ is also NP-hard.*

The following assertion was proved in [8].

Lemma 2.3 *Let a digraph H be obtained from \vec{C}_k , $k \geq 3$, by adding at least one loop. Then $MinHOM(H)$ is NP-hard.*

The following lemma shows that for a digraph H obtained from \vec{C}_3 by adding some loops and backward arcs, i.e., arcs of the form $(i, i - 1)$, $MinHOM(H)$ is NP-hard.

Lemma 2.4 *Let H be a digraph with $V(H) = \{1, 2, 3\}$ and $A(H) = \{12, 23, 32, 31, 22, 33\} \cup B$, where $B \subseteq \{11\}$. Then $MinHOM(H)$ is NP-hard.*

Proof: Let G be a loopless digraph with p vertices. Construct a bipartite digraph D as follows: $V(D) = \{x_1, x_2 : x \in V(G)\}$ and $A(D) = \{x_1x_2 : x \in V(G)\} \cup \{x_2y_1 : xy \in A(G)\}$. Set $c_1(x_1) = 0$, $c_2(x_2) = 3$, $c_2(x_1) = c_1(x_2) = 4p + 1$ and $c_3(x_1) = c_3(x_2) = 2$ for each $x \in V(G)$.

Clearly, $h(x_1) = h(x_2) = 3$ for each $x \in V(D)$ defines a homomorphism h of D to H . Let f be a minimum cost homomorphism of D to H . It follows from the fact that the cost of h is $4p$ that $f(x_2) \neq 1$ and $f(x_1) \neq 2$ for each $x \in V(G)$. Thus, for every arc x_1x_2 of D we have three possibilities of coloring: (a) $f(x_1) = 1, f(x_2) = 2$; (b) $f(x_1) = f(x_2) = 3$; (c) $f(x_1) = 3, f(x_2) = 2$. Because of the three choices and the structure of H , if $f(x_1) = 3$ and $f(x_2) = 2$, we can recolor x_2 so that $f(x_2) = 3$, decreasing the cost of f , a contradiction. Thus, (c) is impossible for f .

Let $f(x_1) = f(y_1) = 1$, where x, y are distinct vertices of G . If $xy \in A(G)$, then $x_2y_1 \in A(D)$, which is a contradiction since $f(x_2) = 2$. Thus, x and y are non-adjacent in G . Hence, $I = \{x \in V(G) : f(x_1) = 1\}$ is an independent set in G . Observe that the cost of f is $4p - |I|$.

Conversely, if I is an independent set in G , we obtain a homomorphism g of D to H by fixing $g(x_1) = 1, g(x_2) = 2$ for $x \in I$ and $g(x_1) = g(x_2) = 3$ for $x \in V(G) - I$. Observe that the cost of g is $4p - |I|$. Hence a homomorphism g of D to H is of minimum cost if and only if the corresponding independent set I is of maximum size in G . Since the maximum size independent set problem is NP-hard, $\text{MinHOM}(H)$ is NP-hard as well. Observe that the validity of the proof does not depend on whether vertex 1 has a loop or not. \diamond

Corollary 2.5 *Let H be a reflexive semicomplete digraph. If H contains either R or \vec{C}_3^* as an induced subdigraph, $\text{MinHOM}(H)$ is NP-hard.*

Proof: It is straightforward to see that Lemmas 2.3 and 2.4 imply the NP-hardness of $\text{MinHOM}(\vec{C}_3^*)$ and $\text{MinHOM}(R)$, respectively. The above statement follows directly from Lemma 2.2. \diamond

The following theorem is from [7].

Theorem 2.6 *Let H be a reflexive graph. If H is a proper interval graph (possibly with more than one component), then the problem $\text{MinHOM}(H)$ is polynomial time solvable. In all other cases, the problem $\text{MinHOM}(H)$ is NP-hard.*

Suppose that H is a semicomplete digraph and that $U(H^{\text{sym}})$ is not a proper interval graph. That is, at least one component of $U(H^{\text{sym}})$ is not a proper interval graph. Then $\text{MinHOM}(U(H^{\text{sym}}))$ is polynomial time reducible to $\text{MinHOM}(H)$ since an input graph G of $\text{MinHOM}(U(H^{\text{sym}}))$ can be transformed into an input digraph G^* of $\text{MinHOM}(H)$ by replacing each edge xy of G by a symmetric arc xy of G^* . Hence, if $U(H^{\text{sym}})$ is not a proper interval graph, $\text{MinHOM}(H)$ is NP-hard by Theorem 2.6. Together with Corollary 2.5, this proves the claim for the NP-hardness part of Theorem 2.1.

Theorem 2.7 *Let H be a reflexive semicomplete digraph. If H contains either R or \vec{C}_3^* as an induced subdigraph, or $U(H^{\text{sym}})$ is not a proper interval graph, then $\text{MinHOM}(H)$ is NP-hard.*

2.2 Polynomial time solvable cases of $\text{MinHOM}(H)$

Let H be a digraph and let v_1, v_2, \dots, v_p be an ordering of $V(H)$. Let $e = v_i v_r$ and $f = v_j v_s$ be two arcs in H . The pair $v_{\min\{i,j\}} v_{\min\{s,r\}} (v_{\max\{i,j\}} v_{\max\{s,r\}})$ is called the

minimum (maximum) of the pair e, f . (The minimum (maximum) of two arcs is not necessarily an arc.) An ordering v_1, v_2, \dots, v_p is a *Min-Max ordering* of $V(H)$ if both minimum and maximum of every two arcs in H are in $A(H)$. Two arcs $e, f \in A(H)$ are called a *non-trivial pair* if $\{e, f\} \neq \{g', g''\}$, where g' (g'') is the minimum (maximum) of e, f . Clearly, to check that an ordering is Min-Max, it suffices to verify that the minimum and maximum of every non-trivial pair of arcs are arcs, too.

The following theorem was proved in [9] for loopless digraphs. In fact, the same proof is valid for digraphs with possible loops.

Theorem 2.8 *Let H be a digraph and let an ordering $1, 2, \dots, p$ of $V(H)$ be a Min-Max ordering, i.e., for any pair ik, js of arcs in H , we have $\min\{i, j\} \min\{k, s\} \in A(H)$ and $\max\{i, j\} \max\{k, s\} \in A(H)$. Then $\text{MinHOM}(H)$ is polynomial time solvable.*

In this subsection, we assume that H is a reflexive semicomplete digraph which contains neither R nor \vec{C}_3^* , and for which $U(H^{\text{sym}})$ is a proper interval graph (possibly with more than one component), unless we mention otherwise. In this subsection, we will show that H has a Min-Max ordering, and, thus, $\text{MinHOM}(H)$ is polynomial time solvable by Theorem 2.8.

There is a useful characterization of proper interval graphs [15, 21].

Theorem 2.9 *A reflexive graph H is a proper interval graph if and only if its vertices can be ordered v_1, v_2, \dots, v_n so that $i < j < k$ and $v_i v_k \in E(H)$ imply that $v_i v_j \in E(H)$ and $v_j v_k \in E(H)$.*

Let H be a digraph and let v_1, v_2, \dots, v_p be an ordering of $V(H)$. We call $v_i v_j$ a *forward arc* (with respect to the ordering) if $i < j$, and a *backward arc* if $i > j$. The following lemma shows that if H satisfies a certain condition, then the vertices of H can be ordered so that every arc is either forward or symmetric.

Lemma 2.10 *Let H be a reflexive semicomplete digraph and suppose H does not contain R as an induced subdigraph and suppose that $U(H^{\text{sym}})$ is a connected proper interval graph. Then the vertices of H can be ordered v_1, v_2, \dots, v_n such that $i < j < k$ and $v_i v_k \in A(H^{\text{sym}})$ imply that $v_i v_j \in A(H^{\text{sym}})$ and $v_j v_k \in A(H^{\text{sym}})$ and furthermore, for every pair of vertices v_i and v_j with $i < j$, we have $v_i \rightarrow v_j$.*

Proof: Since $U(H^{\text{sym}})$ is a proper interval graph, the vertices of H can be ordered v_1, \dots, v_n such that $i < j < k$ and $v_i v_k \in A(H^{\text{sym}})$ imply that $v_i v_j \in A(H^{\text{sym}})$ and $v_j v_k \in A(H^{\text{sym}})$ by Theorem 2.9. Observe that if $v_i v_j$ is a symmetric arc with $i < j$, then for each ℓ, k with $i < \ell < k < j$ we have $v_\ell v_k$ is a symmetric arc. Note also that

$v_i v_{i+1}$ for each $i = 1, \dots, n-1$ is a symmetric arc, since otherwise H^{sym} has more than one component, contradicting the connectivity assumption.

We wish to prove that if $v_\ell \rightarrow v_k$ for some $\ell < k$, then $v_i \rightarrow v_j$ for each $i < j$. We prove it using a sequence of claims.

Claim 1. If $v_i \mapsto v_j$ for some $j > i$, then $v_i \rightarrow \{v_{i+1}, \dots, v_n\}$.

Proof: If $v_i v_k \in A(H^{sym})$ for each $k > i$, there is nothing to prove. Thus, we may assume without loss of generality that there exists a vertex v_k such that $v_k \mapsto v_i$. By an observation above, all arcs between v_i and v_t for each $t \geq \min\{j, k\}$ are asymmetric. Thus, there is an index $m \geq \min\{j, k\}$ such that either $v_i \mapsto v_m$ and $v_{m+1} \mapsto v_i$ or $v_m \mapsto v_i$ and $v_i \mapsto v_{m+1}$. Recall that $v_m v_{m+1}$ is a symmetric arc. Hence, $H[\{v_i, v_m, v_{m+1}\}] \cong R$, a contradiction.

A similar argument leads to the symmetric statement below.

Claim 1'. If $v_j \mapsto v_i$ for some $j < i$, then $\{v_1, \dots, v_{i-1}\} \rightarrow v_i$.

Claim 2. We have either $\{v_1, \dots, v_{i-1}\} \rightarrow v_i \rightarrow \{v_{i+1}, \dots, v_n\}$ or $\{v_{i+1}, \dots, v_n\} \rightarrow v_i \rightarrow \{v_1, \dots, v_{i-1}\}$.

Proof: Suppose to the contrary that there are two vertices v_j, v_k with $j < i < k$ such that $v_i \mapsto v_j$, $v_i \mapsto v_k$ in H . (The case for which $v_j \mapsto v_i$, $v_k \mapsto v_i$ in H can be treated in a similar manner.) Then $v_j v_k$ is not a symmetric arc since otherwise, $v_j v_i$ and $v_i v_k$ must be symmetric arcs by the property of the ordering. Hence, only one of $v_j v_k$ and $v_k v_j$ is an arc of H . In either case, we have a contradiction by Claims 1 or 1'.

Claim 3: If $v_\ell \rightarrow v_k$ for some $\ell < k$, then $v_i \rightarrow v_j$ for each $i < j$.

Proof: Suppose to the contrary that there exist two vertices v_j and v_i such that $v_j \mapsto v_i$ and $i < j$. If any two of the four vertices v_i, v_j, v_ℓ and v_k are identical, we have a contradiction by Claim 1, 1' or 2. Thus, we may assume that these vertices are all distinct. We have the following cases.

(a) Let $i < \ell$. Then we have $i < \ell < k$. If $v_i v_k$ is a symmetric arc, the arc $v_\ell v_k$ must be symmetric by the property of the ordering, a contradiction. Hence, only one of $v_i v_k$ and $v_k v_i$ is an arc of H . If $v_i \mapsto v_k$, we have a contradiction by Claim 1 for vertex v_i and if $v_k \mapsto v_i$, we have a contradiction by Claim 1' for vertex v_k .

(b) Let $\ell < i$. Then we have $\ell < i < j$. If $v_\ell v_j$ is a symmetric arc, then $v_i v_j$ must be a symmetric arc by the property of the ordering, contradiction. Hence, only one of $v_j v_\ell$ and $v_\ell v_j$ is an arc of H . In either case, we have a contradiction by Claims 1 or 1'.

By Claim 3, either v_1, v_2, \dots, v_n or its reversal satisfies the required property. \diamond

Consider a reflexive semicomplete digraph H . Suppose that H does not contain either R or \vec{C}_3^* as an induced subdigraph and $U(H^{sym})$ is a proper interval graph. Note that each isolated vertex in $U(H^{sym})$ forms a trivial proper interval graph in itself.

Suppose that H^{sym} is not connected. If each component of H^{sym} is trivial, it is clear that H has a Min-Max ordering since H is a reflexive transitive tournament (H does not contain \vec{C}_3^*). Hence we may assume that at least one component of H^{sym} is nontrivial. Let H_i^{sym} and H_j^{sym} be two distinct components of H^{sym} and at least one of them, say H_j^{sym} , is a nontrivial component containing more than one vertex. Clearly, the arcs between H_i^{sym} and H_j^{sym} are all asymmetric.

Let u be a vertex of H_i^{sym} , and let v and w be two distinct vertices in H_j^{sym} . Without loss of generality, we may assume that $u \mapsto v$. If $w \mapsto u$, there must exist adjacent vertices p and q on a path from v to w in H_j^{sym} such that $u \mapsto p$ and $q \mapsto u$. Then we have $H[\{u, p, q\}] \cong R$, a contradiction. With a similar argument, it is easy to see that all arcs between two components H_i^{sym} and H_j^{sym} are oriented in the same direction with respect to the components. Furthermore, since H is \vec{C}_3^* -free, the components of H^{sym} can be ordered $H_1^{sym}, H_2^{sym}, \dots, H_l^{sym}$ so that for each pair of vertices $u \in H_i^{sym}$ and $v \in H_j^{sym}$ with $i < j$, we have $u \mapsto v$. This implies the following:

Corollary 2.11 *Let H be a reflexive semicomplete digraph and suppose H does not contain either R or \vec{C}_3^* as an induced subdigraph and suppose that $U(H^{sym})$ is a proper interval graph. Then the components of H^{sym} can be ordered $H_1^{sym}, H_2^{sym}, \dots, H_l^{sym}$ such that if $u \in H_i^{sym}$, $v \in H_j^{sym}$ and $i < j$, then we have $u \mapsto v$.*

We shall call the ordering of the components of H^{sym} described in Corollary 2.11 an *acyclic ordering* of the the components of H^{sym} . Now, with Lemma 2.10, we have the following lemma.

Lemma 2.12 *Let H be a reflexive semicomplete digraph and suppose H does not contain either R or \vec{C}_3^* as an induced subdigraph and $U(H^{sym})$ is a proper interval graph. Then the vertices of H can be ordered v_1, \dots, v_n so that $i < j < k$ and $v_i v_k \in A(H^{sym})$ imply that $v_i v_j \in A(H^{sym})$ and $v_j v_k \in A(H^{sym})$ and furthermore, for every pair of vertices v_i and v_j from $V(H)$ with $i < j$, we have $v_i \rightarrow v_j$.*

Proof: Let $H_1^{sym}, H_2^{sym}, \dots, H_l^{sym}$ be the acyclic ordering of the components of H^{sym} . By Lemma 2.10, we have an ordering $v_1^i, v_2^i, \dots, v_{|V(H_i^{sym})|}^i$ of $V(H_i^{sym})$, for each $i = 1, \dots, l$, such that every asymmetric arc is forward. Then the ordering

$$v_1^1, v_2^1, \dots, v_{|V(H_1^{sym})|}^1, v_1^2, v_2^2, \dots, v_{|V(H_2^{sym})|}^2, \dots, v_1^l, v_2^l, \dots, v_{|V(H_l^{sym})|}^l$$

of the vertices of H satisfies the condition, completing the proof. \diamond

The following proposition was proved in [7]. Observe that for the symmetric subdigraph H^{sym} , the ordering of the vertices of H described in Lemma 2.12 satisfies the condition of the proposition below.

Proposition 2.13 *A reflexive graph H has a Min-Max ordering if and only if its vertices can be ordered v_1, v_2, \dots, v_n so that $i < j < k$ and $v_i v_k \in E(H)$ imply that $v_i v_j \in E(H)$ and $v_j v_k \in E(H)$.*

Lemma 2.14 *Let H be a reflexive semicomplete digraph. If H does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(H^{sym})$ is a proper interval graph, then H has a Min-Max ordering.*

Proof: Let v_1, \dots, v_n be the ordering of $V(H)$ as described in Lemma 2.12. We will show that this is a Min-Max ordering of $V(H)$.

Let $v_i v_j$ and $v_k v_l$ be a non-trivial pair of H . If $i \leq j$ and $k \leq l$, it is easy to see that both the minimum and maximum of $v_i v_j$ and $v_k v_l$ are in $A(H)$. If $i > j$ and $k > l$, $v_i v_j$ and $v_k v_l$ are symmetric arcs of H^{sym} . Since they are a non-trivial pair, the vertices v_i, v_j, v_k and v_l belong to the same component of H^{sym} by the proof of Lemma 2.12. Then the minimum and the maximum of $v_i v_j$ and $v_k v_l$ are also in $A(H^{sym})$ by Lemma 2.10.

Now suppose that $i \leq j$ and $k > l$. Note that $v_k v_l$ is a symmetric arc. Hence if $i = j$, then the vertices v_i, v_k and v_l belong to the same component of H^{sym} , and, thus, the minimum and the maximum of $v_i v_j$ and $v_k v_l$ are in $A(H^{sym})$ by Lemma 2.10. If $i \neq j$, we need to consider the following four cases covering all possibilities for non-trivial pairs:

(a) $i \leq l < j \leq k$. Then $v_{\min\{i,k\}} v_{\min\{j,l\}} = v_i v_l \in A(H)$ as $i \leq l$. Also, since $v_l v_k$ is a symmetric arc, $v_{\max\{i,k\}} v_{\max\{j,l\}} = v_k v_j$ is a symmetric arc by Lemma 2.12.

(b) $l < i < j \leq k$. Then, since $v_l v_k$ is a symmetric arc, $v_{\max\{i,k\}} v_{\max\{j,l\}} = v_k v_j$ and $v_{\min\{i,k\}} v_{\min\{j,l\}} = v_i v_l$ are symmetric arcs by Lemma 2.12.

(c) $l < i < k < j$. Then, $v_{\max\{i,k\}} v_{\max\{j,l\}} = v_k v_j \in A(H)$ as $k < j$. Also, since $v_l v_k$ is a symmetric arc, $v_{\min\{i,k\}} v_{\min\{j,l\}} = v_i v_l$ is a symmetric arc by Lemma 2.12.

(d) $i \leq l < k < j$. Then $v_{\min\{i,k\}} v_{\min\{j,l\}} = v_i v_l \in A(H)$ and $v_{\max\{i,k\}} v_{\max\{j,l\}} = v_k v_j \in A(H)$ as $i \leq l$ and $k < j$. \diamond

Theorem 2.15 *Let H be a reflexive semicomplete digraph. If H does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(H^{sym})$ is a proper interval graph, then $MinHOM(H)$ is polynomial time solvable.*

Proof: This is a direct consequence of Lemmas 2.14 and 2.8. \diamond

Corollary 2.16 *Suppose $P \neq NP$. Let H be a reflexive semicomplete digraph. Then $MinHOM(H)$ is polynomial time solvable if and only if H has a Min-Max ordering.*

Proof: This is a direct consequence of Lemma 2.14 and Theorem 2.7.

3 Classification for Semicomplete Digraphs with Possible Loops

In this section, we describe a dichotomy classification for $\text{MinHOM}(H)$ when H is a semicomplete digraph with possible loops. Let W be a digraph with $V(W) = \{1, 2\}$ and $A(W) = \{12, 21, 22\}$. Let R' be a digraph with $V(R') = \{1, 2, 3\}$ and $A(R') = \{12, 23, 32, 31, 22, 33\}$.

Given a semicomplete digraph H w.p.l., let $L = L(H)$ and $I = I(H)$ denote the maximal induced subdigraphs of H which are reflexive and loopless, respectively. When $H = L$, we have obtained a dichotomy classification for reflexive semicomplete digraph in Section 2. When $H = I$, we also have a dichotomy classification by the following theorem from [9].

Theorem 3.1 *For a semicomplete digraph H , $\text{MinHOM}(H)$ is polynomial time solvable if H is acyclic or $H = \vec{C}_k$ for $k=2$ or 3 , and NP-hard, otherwise.*

In this section, we will show that the following dichotomy classification holds when H is a semicomplete digraph w.p.l.

Theorem 3.2 *Let H be a semicomplete digraph with possible loops. If one of the following holds, then $\text{MinHOM}(H)$ is polynomial time solvable. Otherwise, it is NP-hard.*

(i) *The digraph $H = \vec{C}_k$ for $k=2$ or 3 .*

(ii-a) *The digraph L does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(L^{\text{sym}})$ is a proper interval graph; I is a transitive tournament; H does not contain either W , R' or \vec{C}_3 with at least one loop as an induced subdigraph.*

or equivalently,

(ii-b) *The digraph $H = \text{TT}_k[S_1, S_2, \dots, S_k]$ where S_i for each $i = 1, \dots, k$ is either a single vertex without a loop, or a reflexive semicomplete digraph which does not contain R as an induced subdigraph and for which $U(S_i^{\text{sym}})$ is a connected proper interval graph.*

Through subsections 3.1 and 3.2, we will consider only the polynomiality condition (ii-a) in Theorem 3.2. We first prove the NP-hardness part of Theorem 3.2 in subsection 3.1. In subsection 3.2, a proof for the polynomial solvable case is given. Finally in subsection 3.3, we will prove the equivalence of condition (ii-a) and (ii-b) in Theorem 3.2.

3.1 NP-hard cases of $\text{MinHOM}(H)$

The following two lemmas were proved in [8].

Lemma 3.3 *MinHOM(W) is NP-hard.*

Lemma 3.4 *Let H' be a digraph obtained from $\vec{C}_k = 12 \dots k1$, $k \geq 2$, by adding an extra vertex $k+1$ dominated by at least two vertices of the cycle and let H'' is the digraph obtained from H' by adding the loop at vertex $k+1$. Let H be H' or its converse or H'' or its converse. Then $\text{MinHOM}(H)$ is NP-hard.*

Observe that $\text{MinHOM}(R')$ is NP-hard by Lemma 2.4. The following result was proved in [2].

Theorem 3.5 *Let H be a (loopless) semicomplete digraph with at least two directed cycles. Then the problem of checking whether a digraph D has an H -coloring is NP-complete.*

Lemma 3.6 *Let H be a digraph with $V(H) = \{1, 2, 3\}$ and $A(H) = \{12, 21, 23, 31, 33\}$. Then $\text{MinHOM}(H)$ is NP-hard.*

Proof: We will reduce the maximum independent set problem to $\text{MinHOM}(H)$. Before we do this we consider a digraph $D^*(u, v)$ defined as follows. Here we set $e = uv$:

$$V(D^*(u, v)) = \{u^e, u, v^e, v, x_1^e, x_2^e, \dots, x_6^e\}$$

$$A(D^*(u, v)) = \{x_1^e x_2^e, x_2^e x_3^e, \dots, x_5^e x_6^e, x_6^e x_1^e, x_4^e u^e, u^e u, x_5^e v^e, v^e v\}$$

Let G be a graph with p vertices. Construct a digraph D as follows: Start with $V(D) = V(G)$ and, for each edge $e = uv \in E(G)$, add a distinct copy of $D^*(u, v)$ to D . Note that the vertices in $V(G)$ form an independent set in D and that $|V(D)| = |V(G)| + 8|E(G)|$.

Given an edge $e = uv \in E(G)$, we fix the costs as follows: Let $c_1(x_1^e) = 0$ and $c_i(x_i^e) = p + 1$ for each $i = 2, 3$. Let $c_i(x_j^e) = 0$ for each $i = 1, 2, 3$ and $j = 2, \dots, 6$ apart from $c_3(x_4^e) = c_3(x_5^e) = p + 1$. Also, $c_i(u^e) = c_i(v^e) = 0$ for each $i = 1, 2, 3$, $c_2(u) = c_2(v) = 0$, $c_1(u) = c_1(v) = 1$ and $c_3(u) = c_3(v) = p + 1$.

Consider a mapping h of $V(D)$ to $V(H)$ as follows: $h(x_i^e) = 1$ if i is odd, $h(x_i^e) = 2$ if i is even, $h(u^e) = 3$, $h(v^e) = 2$ for each $e \in E(G)$ and $h(u) = 1$ for each $u \in V(G)$. It is easy to check that h defines a homomorphism of D to H and the cost of h is p . Let f be a minimum cost homomorphism of D to H . It follows from the fact that the cost of h is p that $f(x_1^e) = 1$, $f(x_4^e), f(x_5^e) \in \{1, 2\}$ for each $e \in E(G)$, and $f(u) \in \{1, 2\}$ for each $u \in V(G)$. Moreover, due to the structure of $D^*(u, v)$ and the costs, for each $e \in E(G)$, $(f(x_1^e), \dots, f(x_6^e))$ must coincide with one of the following two sequences: $(1, 2, 1, 2, 1, 2)$ or $(1, 2, 3, 1, 2, 3)$.

If the first sequence is the actual one, then we have $f(x_4^e) = 2$, $f(u^e) \in \{1, 3\}$, $f(u) \in \{1, 2\}$ and $f(x_5^e) = 1$, $f(v^e) = 2$, $f(v) = 1$. If the second sequence is the actual one, we

have $f(x_4^e) = 1$, $f(u^e) = 2$, $f(v) = 1$ and $f(x_5^e) = 2$, $f(v^e) \in \{1, 3\}$, $f(v) \in \{1, 2\}$. So in both cases we can assign both u and v color 1. Furthermore, by choosing the right sequence we can color one of u and v with color 2 and the other with color 1. Notice that f cannot assign color 2 to both u and v .

Clearly, f must assign as many vertices of $V(G)$ in D color 2. However, if uv is an edge in G , by the argument above, f cannot assign color 2 to both u and v . Hence, $I = \{u \in V(G) : f(u) = 2\}$ is an independent set in G . Observe that the cost of f is $p - |I|$.

Conversely, if I' is an independent set in G , we obtain a homomorphism g of D to H by fixing $g(u) = 2$ for $u \in I'$, $g(u) = 1$ for $u \notin I'$. We can choose an appropriate sequence for x_1^e, \dots, x_6^e for each edge $e \in E(G)$ and fix the assignment of u^e and v^e accordingly by the above argument. Observe that the cost of g is $p - |I'|$. Hence the cost of a minimum homomorphism f of D to H is $p - \alpha$, where α is the size of the maximum independent set in G . Since the maximum size independent set problem is NP-hard, $\text{MinHOM}(H)$ is NP-hard as well. \diamond

Lemma 3.7 *Let H be a digraph with $V(H) = \{1, 2, 3\}$ and $A(H) = \{12, 21, 23, 31\} \cup B_1 \cup B_2$, where B_1 is either $\{11, 22\}$ or \emptyset and B_2 is either $\{33\}$ or \emptyset . Then $\text{MinHOM}(H)$ is NP-hard.*

Proof: Consider the following three cases.

Case 1: $B_1 = \emptyset$ and $B_2 = \emptyset$. Then $\text{MinHOM}(H)$ is NP-hard by Theorem 3.5.

Case 2: $B_1 = \emptyset$ and $B_2 = \{33\}$. Then $\text{MinHOM}(H)$ is NP-hard by Lemma 3.6

Case 3: $B_1 = \{11, 22\}$. Then $\text{MinHOM}(H)$ is NP-hard by Lemma 2.4. \diamond

Consider a strong semicomplete digraph w.p.l. H on three vertices. We want to obtain all polynomial cases for H . If H does not have a 2-cycle, $\text{MinHOM}(H)$ is NP-hard if (and only if) at least one of its vertices has a loop by Lemma 2.3. Note that we have a polynomial case if H is \vec{C}_3 . Suppose that H has at least one 2-cycle. If there are two or more 2-cycles, $\text{MinHOM}(H)$ is NP-hard by Theorem 3.5 and Lemma 3.3 unless H is reflexive. Note that in the reflexive case, $\text{MinHOM}(H)$ is polynomial time solvable since H has a Min-Max ordering.

Now suppose that H has only one 2-cycle. For $\text{MinHOM}(H)$ to be not NP-hard, both or neither of the two vertices forming the 2-cycle must have loops simultaneously since otherwise, $\text{MinHOM}(H)$ is NP-hard by Lemma 3.3. Now by Lemma 3.7, $\text{MinHOM}(H)$ is still NP-hard.

Let $K_3^* - e$ be a digraph obtained by removing a nonloop arc from K_3^* . The above observation can be summarized by the following statement.

Corollary 3.8 *Let H be a strong semicomplete digraph w.p.l. on three vertices. If H is either \vec{C}_3 , K_3^* or $K_3^* - e$, $\text{MinHOM}(H)$ is polynomial time solvable. Otherwise, $\text{MinHOM}(H)$ is NP-hard.*

For a semicomplete digraph w.p.l. H , if either $\text{MinHOM}(L)$ or $\text{MinHOM}(I)$ is NP-hard, $\text{MinHOM}(H)$ is NP-hard by Lemma 2.2. (Recall that $L = L(H)$ and $I = I(H)$ denote the maximal induced subdigraphs of H which are reflexive and loopless, respectively.) Also, if H contains \vec{C}_3 with at least one loop as an induced subdigraph, $\text{MinHOM}(H)$ is NP-hard by Lemmas 2.3 and 2.2. Suppose $\text{MinHOM}(H)$ is not NP-hard. Then Lemma 3.3 indicates that for any pair of vertices $u \in V(L)$ and $v \in V(I)$, either $u \rightarrow v$ or $v \rightarrow u$, not both, as otherwise $\text{MinHOM}(H)$ is NP-hard. With these observations and Lemma 3.4, the following statement is easily derived.

Lemma 3.9 *Let H be a semicomplete digraph. If one of the following condition holds, $\text{MinHOM}(H)$ is NP-hard.*

- (a) I contains a cycle and $I \neq \vec{C}_k$ for $k = 2$ or 3 .
- (b) L contains either R or \vec{C}_3^* as an induced subdigraph, or $U(L^{sym})$ is not a proper interval graph.
- (c) $I = \vec{C}_k$ for $k = 2$ or 3 , and L is nonempty.
- (d) H contains W , R' or \vec{C}_3 with at least one loop as an induced subdigraph.

Proof: If condition (a) holds, $\text{MinHOM}(H)$ is NP-hard by Theorem 3.1 and Lemma 2.2. If condition (b) holds, $\text{MinHOM}(H)$ is NP-hard by Theorem 2.1 and Lemma 2.2. If condition (d) holds, $\text{MinHOM}(H)$ is NP-hard by Lemmas 3.3, 2.4, 2.3 and 2.2. The only remaining part is to prove that the condition (c) is sufficient for $\text{MinHOM}(H)$ to be NP-hard.

If $I = \vec{C}_2$ and u is a vertex of L , then we may assume that either u dominates both vertices of I , or u is dominated by one of $V(I)$ and dominates the other without loss of generality. In the former case, $\text{MinHOM}(H)$ is NP-hard by Lemma 3.4. In the latter case, $\text{MinHOM}(H)$ is NP-hard by Lemma 3.6. If $I = \vec{C}_3$ and u is a vertex with a loop, $\text{MinHOM}(H)$ is NP-hard by Lemma 3.4. \diamond

In fact, Lemma 3.9 proves the NP-hardness part in Theorem 3.2. This can be seen as follows. Suppose that $\text{MinHOM}(H)$ is not NP-hard. Recall that the polynomiality conditions of Theorem 3.2 are: (i) $H = \vec{C}_k$ for $k = 2$ or 3 , or (ii-a) L does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph, I is a transitive tournament, and H does not contain either W , R' or \vec{C}_3 with at least one loop as an induced subdigraph.

Suppose that a semicomplete digraph w.p.l. H has an loopless cycle. Then condition (ii-a) does not hold, and for condition (i) to be violated, either one of (a) and (c) in Lemma

3.9 must hold. On the other hand, suppose that the loopless part I of H is a transitive tournament. Then condition (i) does not hold, and for condition (ii-a) to be violated, one of (b) and (d) in Lemma 3.9 must hold.

Corollary 3.10 *Let H be a semicomplete digraph with possible loops. If none of the following holds, then $\text{MinHOM}(H)$ is NP-hard.*

(a) *The digraph $H = \vec{C}_k$ for $k = 2$ or 3 .*

(b) *The digraph L does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(L^{\text{sym}})$ is a proper interval graph; I is a transitive tournament; H does not contain either W , R' or \vec{C}_3 with at least one loop as an induced subdigraph.*

3.2 Polynomial time solvable cases of $\text{MinHOM}(H)$

If condition (i) in Theorem 3.2 holds for a semicomplete digraph w.p.l. H , $\text{MinHOM}(H)$ is clearly polynomial time solvable by Theorem 3.1. Although \vec{C}_3 does not have a Min-Max ordering, there is a simple algorithm which solves $\text{MinHOM}(H)$ in polynomial time when $H = \vec{C}_k$, $k \geq 2$, see [9, 8].

Therefore, we only need to prove that when H satisfies the condition (ii-a) in Theorem 3.2, $\text{MinHOM}(H)$ is polynomial time solvable. We claim that H has a Min-Max ordering in this case. Before showing this claim, we prove that the ordering described in Lemma 2.12 for a reflexive semicomplete digraph can be extended to a semicomplete digraph w.p.l. if condition (ii-a) in Theorem 3.2 is satisfied.

Lemma 3.11 *Let H be a semicomplete digraph with possible loops. Suppose that L does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(L^{\text{sym}})$ is a proper interval graph. Also suppose that I is a transitive tournament and H does not contain either W , R' or \vec{C}_3 with at least one loop as an induced subdigraph. Then the vertices of H can be ordered v_1, \dots, v_n so that for every pair of vertices v_i and v_j with $i < j$, we have $v_i \rightarrow v_j$.*

Proof: Let $L_1^{\text{sym}}, \dots, L_l^{\text{sym}}$ be the acyclic ordering of the components of L^{sym} . Let

$$w_1, w_2, \dots, w_q = v_1^1, v_2^1, \dots, v_{|V(L_1^{\text{sym}})|}^1, \dots, v_1^i, v_2^i, \dots, v_{|V(L_i^{\text{sym}})|}^i, \dots, v_1^l, v_2^l, \dots, v_{|V(L_l^{\text{sym}})|}^l$$

be the ordering of $V(L)$ as described in Lemma 2.12. Let u_1, \dots, u_p be the acyclic ordering of $V(I)$, i.e., $u_i \rightarrow u_j$ implies $i < j$. We will prove the statement by showing that the subdigraph induced by $V(L_i^{\text{sym}})$ can be 'inserted' into an appropriate position among the acyclic ordering of $V(I)$ without creating a cycle, thus by constructing an ordering of $V(H)$ satisfying the asserted property.

Observe that any arc between a pair of two vertices from I and L is asymmetric. Otherwise, the two vertices induce a digraph W , which is impossible by the assumption.

First, we claim that given a vertex u of I and a component L_i^{sym} , we have either $u \mapsto V(L_i^{sym})$ or $V(L_i^{sym}) \mapsto u$. If L_i^{sym} is a trivial component consisting of a single vertex, the claim follows directly. So, assume that $|V(L_i^{sym})| \geq 2$ and there exists two vertices v, v' of L_i^{sym} such that $u \mapsto v$ and $v' \mapsto u$. Then, since L_i^{sym} is connected, there is a path in L_i^{sym} linking v and v' . We can find two adjacent vertices s, t on this path such that $u \mapsto s$ and $t \mapsto u$. However, then $H[\{u, s, t\}] \cong R'$, a contradiction.

Secondly, we claim that for each component L_i^{sym} (possibly trivial), the vertices of L_i^{sym} can be ‘inserted’ into an appropriate position so that the ordering of $V(I) \cup V(L_i^{sym})$ satisfies the required property. (Here, the ordering within $V(L_i^{sym})$ remains unchanged.) That is, either $V(I) \mapsto V(L_i^{sym})$ or $V(L_i^{sym}) \mapsto V(I)$, or there exist an integer $1 \leq j < p$ such that for all $k \leq j$, we have $u_k \mapsto V(L_i^{sym})$ and for all $k > j$, we have $V(L_i^{sym}) \mapsto u_k$. If $V(I) \mapsto V(L_i^{sym})$ or $V(L_i^{sym}) \mapsto V(I)$, the ordering u_1, \dots, u_p followed by or following the ordering of $V(L_i^{sym})$ trivially satisfies the required property. Thus, we may assume that $u \mapsto V(L_i^{sym})$ and $V(L_i^{sym}) \mapsto u'$ for some $u, u' \in V(I)$.

Suppose that there are two vertices u_j and $u_{j'}$ of I with $j' < j$ such that $u_j \mapsto V(L_i^{sym})$ and $V(L_i^{sym}) \mapsto u_{j'}$. Then $u_{j'}, u_j$ together with a vertex of L_i^{sym} form \vec{C}_3 with a loop, contradicting the assumption. Hence, if $u_j \mapsto V(L_i^{sym})$ for some $u_j \in V(I)$, then $u_{j'} \mapsto V(L_i^{sym})$ for each $j' < j$. Similarly, if $V(L_i^{sym}) \mapsto u_j$ for some $u_j \in V(I)$, then $V(L_i^{sym}) \mapsto u_{j'}$ for each $j' > j$. By taking the maximum j such that $u_j \mapsto V(L_i^{sym})$

and inserting $V(L_i^{sym})$ between u_j and u_{j+1} while preserving the ordering within $V(L_i^{sym})$, we are done with the claim. From now on, we will say that $V(L_i^{sym})$ is *inserted after* u_j if $u_j \mapsto V(L_i^{sym})$ and $V(L_i^{sym}) \mapsto u_{j+1}$ when u_{j+1} exists, and $V(L_i^{sym})$ is *inserted before* u_j if $V(L_i^{sym}) \mapsto u_j$ and $u_{j-1} \mapsto V(L_i^{sym})$ when u_{j-1} exists.

Note that if any two components L_i^{sym} and L_j^{sym} of L^{sym} are inserted before/after the same vertex of I , we will keep their relative order unchanged.

Now let us show that the insertion of all $V(L_i^{sym})$ ’s does not change their relative order in L . That is, if $V(L_i^{sym})$ is inserted after u_j , then each component $L_{i'}^{sym}$ for $i' > i$ is inserted after $u_{j'}$, where $j' \geq j$ and if L_i^{sym} is inserted before u_j , then each component $L_{i'}^{sym}$ for $i' < i$ is inserted before $u_{j'}$, where $j' \leq j$.

Suppose to the contrary that there are two components L_i^{sym} and $L_{i'}^{sym}$ with $i < i'$ such that $V(L_i^{sym})$ is inserted after u_j and $V(L_{i'}^{sym})$ is inserted before $u_{j'}$ with $j' \leq j$. Then, by the above argument, $V(L_{i'}^{sym}) \mapsto u_j$. However, a vertex from $V(L_{i'}^{sym})$, a vertex from $V(L_i^{sym})$ and u_j induce \vec{C}_3 with two loops, contradicting the assumption. Hence, if $V(L_i^{sym})$ is inserted after u_j , then $V(L_{i'}^{sym})$ for $i' > i$ is inserted after $u_{j'}$, where $j' \geq j$. Similarly, we can show that if $V(L_i^{sym})$ is inserted before u_j , then $V(L_{i'}^{sym})$ for $i' < i$ is inserted before $u_{j'}$, where $j' \leq j$.

It is straightforward from the above construction that the resulting ordering satisfies the required property. \diamond

Now we are ready to prove that H has a Min-Max ordering when H satisfies the condition (ii-a) in Theorem 3.2.

Lemma 3.12 *Let H be a semicomplete digraph with possible loops. Suppose that L contains neither R nor \vec{C}_3^* as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph. Also suppose that I is a transitive tournament and H does not contain either W , R' or \vec{C}_3 with at least one loop as an induced subdigraph. Then $MinHOM(H)$ has a Min-Max ordering.*

Proof: Consider an ordering v_1, \dots, v_n of the vertices of H as described in Lemma 3.11. We will show that this is a Min-Max ordering of $V(H)$. Note that the induced ordering of $V(I)$ is an acyclic ordering and the induced ordering of $V(L)$ is a Min-Max ordering for L^{sym} as described in Lemma 2.12.

Let $v_i v_j$ and $v_k v_l$ be any nontrivial pair of arcs of H . Observe that if both arcs are in $A(L)$, then the minimum and the maximum of them are also in $A(L)$ since the induced ordering of $V(L)$ is a Min-Max ordering for L . Moreover, if both arcs are forward arcs, i.e. $i < j$ and $k < l$, then we have either $i < k < l < j$ or $k < i < j < l$. In either case, it follows from Lemma 3.11 that the minimum and the maximum of them are in $A(H)$.

Hence what we need to consider is the case where $v_k v_l$ is not a forward arc. If $v_k v_l$ is a loop, then $i < k = l < j$. It follows from Lemma 3.11 that the minimum and the maximum of the two arcs are in $A(H)$ in this case. Let $v_k v_l$ be a backward arc, i.e., $k > l$. Clearly, $v_k v_l \in A(L)$. Then there are two remaining cases to consider.

Case 1: $v_i v_j \in A(I)$.

Then we have one of the following options: (a) $i < l < j < k$, (b) $i < l < k < j$, (c) $l < i < k < j$, (d) $l < i < j < k$. However, in (a), $v_l \mapsto v_j$ and $v_j \mapsto v_k$, which is a contradiction since v_k, v_l belong to the same component of L^{sym} , and v_j has to either dominate or to be dominated by each component of L^{sym} . With a similar argument, case (c) and case (d) are impossible. By Lemma 3.11, in case (b), the minimum and the maximum of $v_i v_j$ and $v_k v_l$ are in $A(H)$.

Case 2: $v_i v_j \in A(H) \setminus (A(I) \cup A(L))$.

Since $v_i v_j \in A(H) \setminus (A(I) \cup A(L))$, exactly one of v_i and v_j has a loop. Assume that v_j has a loop. The case for which v_i has a loop can be treated in a similar manner.

Then we have one of the following options: (a) $i < l < j \leq k$, (b) $i < l < k < j$, (c) $l < i < k \leq j$, (d) $l < i < j < k$. However, if (c) is the case, $v_l \rightarrow v_i$ and $v_i \rightarrow v_k$, which is a contradiction since v_k, v_l belong to the same component of L^{sym} and v_i has to either dominate or be dominated by each component of L^{sym} . With a similar argument, case (d) is impossible.

Let (a) be the case. Note that v_k and v_l belong to the same component of L^{sym} .

By the property of the ordering (see the proof of Lemma 3.11), v_j belongs to the same component of L^{sym} with v_k and v_l . Since the ordering of the vertices in this component satisfies the condition in Lemma 2.9, $v_l v_k \in A(L^{sym})$ and $l < j \leq k$ imply that $v_k v_j = v_{\max\{i,k\}} v_{\max\{j,l\}} \in A(L^{sym})$. By Lemma 3.11, $v_i v_l = v_{\min\{i,k\}} v_{\min\{j,l\}} \in A(H)$.

Let (b) be the case. By Lemma 3.11, both the minimum and the maximum of the two arcs are in $A(H)$. \diamond

Theorem 3.13 *Let H be a semicomplete digraph with possible loops. If one of the followings holds, then $MinHOM(H)$ is polynomial time solvable.*

(a) *The digraph $H = \vec{C}_k$ for $k = 2$ or 3 .*

(b) *The digraph L does not contains either R or \vec{C}_3^* as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph; I is a transitive tournament; H does not contain either W , R' or \vec{C}_3 with at least one loop as an induced subdigraph.*

Proof: Consider the following cases.

Case 1: The condition (a) holds. Then, there is a polynomial time algorithm for $MinHOM(H)$. We give the algorithm for the sake of completeness. We consider $H = \vec{C}_k$ with an arbitrary integer $k \geq 2$. We assume that the input digraph D is connected since otherwise, the algorithm can be applied to each component of D and we can sum up the costs of homomorphisms of each component to H .

Choose a vertex x of D , and assign it color 1. For any vertex y with color i , we assign all the in-neighbors of y color $i - 1$ and all the out-neighbors of y color $i + 1$, where the operation is taken modulo k . It is easy to see that no vertex of D is assigned a pair of conflicting colors if and only if D has a \vec{C}_k -coloring. Furthermore, cyclicly permutating the colors of $V(D)$ does not affect the existence of a homomorphism of D to H . Hence, we can assign x color $2, \dots, k$, modify the assignment of other vertices of D accordingly, and compute the cost of homomorphism respectively. We finally accept an assignment which leads to the minimum cost.

Case 2: The condition (b) holds. Then by Lemma 3.12 and Theorem 2.8, $MinHOM(H)$ is polynomial time solvable. \diamond

Corollary 3.14 *Let H be a semicomplete digraph w.p.l. Then $MinHOM(H)$ is polynomial time solvable if $H = \vec{C}_k$ for $k = 2$ or 3 , or H has a Min-Max ordering. Otherwise, $MinHOM(H)$ is NP-hard.*

3.3 Proving that (ii-a) and (ii-b) of Theorem 3.2 are equivalent

In this subsection, we will prove that (ii-a) and (ii-b) in Theorem 3.2 are equivalent.

It follows from the proof of Lemma 3.11 that the condition (ii-a) implies (ii-b). Indeed, from the construction of the ordering in the proof of Lemma 3.11, H is a composition digraph, i.e., $H = TT_{p+l}[S_1, S_2, \dots, S_{p+l}]$ where S_i for each $i = 1, \dots, p+l$ is one of the two types: (a) a single vertex without a loop, (b) a reflexive semicomplete digraph which does not contain R as an induced subdigraph, and for which $U(S_i^{sym})$ is a connected proper interval graph. Here, p is the number of vertices in $V(I)$ and l is the number of components (possibly trivial) of L^{sym} .

Lemma 3.16 given below shows that the converse is also true, accomplishing the equivalence of (ii-a) and (ii-b) in Theorem 3.2.

For further reference, we give a well-known theorem that characterizes proper interval graphs in terms of forbidden subgraphs. We will start with some definitions. A graph G is called a *claw* if $V(G) = \{x_1, x_2, x_3, y\}$ and $E(G) = \{x_1y, x_2y, x_3y\}$. A graph G with $V(G) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ is called a *net* if $E(G) = \{x_1x_2, x_2x_3, x_3x_1, y_1x_1, y_2x_2, y_3x_3\}$, and a *tent* if $E(G) = \{x_1x_2, x_2x_3, x_3x_1, y_1x_2, y_1x_3, y_2x_1, y_2x_3, y_3x_1, y_3x_2\}$.

Theorem 3.15 [15] *A graph G is a proper interval graph if and only if it does not contain a cycle of length at least four, a claw, a net, or a tent as an induced subgraph.*

Lemma 3.16 *Let $H = TT_k[S_1, S_2, \dots, S_k]$ where S_i for each $i = 1, \dots, k$ is either a single vertex without a loop, or a reflexive semicomplete digraph which does not contain R as an induced subdigraph and for which $U(S_i^{sym})$ is a connected proper interval graph. Then, H is a semicomplete digraph w.p.l. such that L does not contain either R or \vec{C}_3^* as an induced subdigraph, and $U(L^{sym})$ is a proper interval graph, I is a transitive tournament and H does not contain either W , R' or \vec{C}_3 with at least one loop as an induced subdigraph.*

Proof: Clearly, H is a semicomplete digraph w.p.l. and I is a transitive tournament. Furthermore, the absence of W , R' or \vec{C}_3 with one or two loops in H follows from the transitive tournament structure of H .

Therefore, it remains to show that L does not contain \vec{C}_3^* as an induced subdigraph. To the contrary, suppose that there are vertices $u, v, w \in V(L)$ such that $H[\{u, v, w\}] \cong \vec{C}_3^*$ ($u \mapsto v \mapsto w \mapsto u$). Then u, v and w must belong to the same component S_i . Since S_i^{sym} is connected, there exist paths between any pair of vertices in $\{u, v, w\}$. Define $\mu(u, v, w) = \min\{\text{dist}(u, v), \text{dist}(u, w), \text{dist}(v, w)\}$, where $\text{dist}(x, y)$ is the length of a shortest path between x and y in S_i^{sym} .

Choose a triple u, v, w in S_i such that $H[\{u, v, w\}] \cong \vec{C}_3^*$ ($u \mapsto v \mapsto w \mapsto u$) and $\mu(u, v, w)$ is minimal. Assume that $\text{dist}(u, v) = \mu(u, v, w)$. Consider a shortest path $P = u(= u_0), u_1, \dots, u_p(= v)$ between u and v in S_i^{sym} . Observe that $\text{dist}(u, v) \geq 2$. Let a_v (a_w) be an arc between v and u_1 (between w and u_1). If both a_v and a_w are symmetric, then u, v, w and u_1 form a claw in S_i^{sym} , which is impossible by Theorem 3.15. Hence, at most one of a_v and a_w is symmetric.

If a_v is symmetric, then a_w must be asymmetric and we have either $H[\{u, w, u_1\}] \cong R$ or $H[\{v, w, u_1\}] \cong R$, a contradiction. Similarly, if a_w is symmetric, we have either $H[\{u, v, u_1\}] \cong R$ or $H[\{w, v, u_1\}] \cong R$, also a contradiction. Hence, both a_v and a_w are asymmetric. Suppose that $u_1 \mapsto w$. Then $H[\{u, u_1, w\}] \cong \vec{C}_3^*$, a contradiction. Hence, $w \mapsto u_1$. Similarly, $u_1 \mapsto v$. Thus, u_1, v, w is a triple with $H[\{u_1, v, w\}] \cong \vec{C}_3^*$ such that $\mu(u_1, v, w) < \mu(u, v, w)$, a contradiction to the choice of u, v, w .

Thus, L does not contain \vec{C}_3^* as an induced subdigraph, which completes the proof. \diamond

4 Further Research

We obtained a dichotomy classification for reflexive semicomplete digraphs and semicomplete digraphs w.p.l. This solves the question raised in our previous paper [8]. The obtained results imply that given a (loopless) semicomplete digraph H , for $\text{MinHOM}(H)$ to be polynomial time solvable, H should be a very simple directed cycle or it has to be acyclic. On the other hand, in the dichotomy for reflexive semicomplete digraphs, Min-Max ordering appears to play the central role to characterize the polynomial time solvable cases. We'd like to conjecture that for a reflexive digraph H , $\text{MinHOM}(H)$ is polynomial time solvable if and only if H has a Min-Max ordering. The comparison between this conjecture and the proper ordering conjecture suggested by Feder et. al. in [6] for $\text{ListHOM}(H)$, when H is a reflexive digraph, presents an interesting point of view.

The problem of obtaining a dichotomy classification for semicomplete k -partite digraphs, $k \geq 2$, w.p.l. remains still open. In fact, even settling a dichotomy for k -partite tournaments seems to be not easy. Actually, for a k -partite tournament w.p.l. H , a complete dichotomy of $\text{MinHOM}(H)$ has been obtained in [8] provided that H has a cycle.

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